Alliance Formation in a Multipolar World

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Abstract

We propose a multilayer network approach to alliance formation. In a signed a¢ nity layer, agents are partitioned into clusters, with friendly relations within and hostile connections across clusters. Agents then form defensive collaborations in an alliance layer as follows: Agents in the same cluster form a nested split graph with degree inversely correlated to the level of hostility, and agents from disparate clusters with high-degree and low-hostility form cliques. Within cliques, agents from a cluster that is "intermediate" in terms of discord serve as a bridge to interconnect agents from more "extreme" clusters.

Key words and phrases: Alliance formation, signed graphs, nested split graphs, pairwise stability, cliques.
JEL Classi…cation: C72, D74, D85

1 Introduction

This paper is a contribution to the literature on alliance formation under con‡ict. It explores the incentives of agents (individuals, groups or nations) to form defense alliances when they are embedded in a pre-existing network of bilateral a¢ nities that are friendly, hostile or neutral, and for these a¢ nities in turn to be revised following the formation of defense collaborations. This primitive non-empty network (equivalently *layer* or *graph*) of *a¢ nities* is assumed to be a possible consequence of political, religious, ideological, cultural or historical factors.[1] It can be formally represented as a *signed* network in which a positive link between two

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agents denotes friends, a negative link denotes enemies, and lack of a link denotes a neutral relationship.[2] While there is a large literature on network formation under con‡ict[3], our point of departure is an explicit two-way interaction between the signed network of a¢ nities and the network of defense *alliances* among friends to thwart potential con‡icts with enemies. This interaction is examined through the lens of a *multilayer* network. Figure 1 depicts a multilayer network in which the base layer is the a¢ nity network (denoted by H) and the accompanying layer is the network of defense alliances (denoted by G). The a¢ nity layer is a signed network in which a solid line connecting two agents denotes friendship (a positive relationship), a dashed line denotes hostility (a negative relationship), and lack of a connection denotes a neutral (zero) relationship. The alliance layer is an unsigned network in which a (solid) line connecting two agents denotes a defense collaboration and the absence of a line implies no such collaboration.

Figure 1: A Multilayer Network

Our paper is motivated by the fact that the complex web of interlocking defense alliances that characterize the world today can best be understood as a multilayer network building up from base a¢ nities. The period of the Cold War was characterized by an a¢ nity network in which countries were broadly divided into an Eastern and a Western bloc based on opposing political ideologies. The corresponding alliance network was *bipolar*: the Eastern bloc formed the Warsaw Pact while the Western bloc formed NATO, with no overlap between the two security pacts. The fall of the Berlin Wall altered the a¢ nity network with former Eastern bloc countries recalibrating their relationships with the Western bloc. The resulting alliance network was *unipolar* with former Warsaw Pact members such as Poland, Hungary, Bulgaria,

[2] Signed networks are discussed in Cartwright and Harary (1956), Davis (1967), and Easley and Kleinberg (2010, Chapter 5).

[3] Please see Bloch (2012) and Goyal et al. (2016) for an excellent description of the main lines of research on alliance formation under con‡ict.

Romania and the Czech Republic joining NATO. The current alliance network is sometimes described as *multipolar*, which is inaccurate since nations cannot be divided into mutually exclusive coalitions that jointly coordinate their actions as a set. Instead, we see nations forming alliances across a¢ nities. For example, the Economist[4] has noted that the United States has established bilateral alliances with Australia, Japan, Philippines, South Korea and Thailand in a hub-and-spoke network and quoted the prime minister of Japan, Kishida Fumio, as saying that promoting alliances among the spokes "will lead to the establishment of a *multilayered network*

Second, under what circumstances will agents have an incentive to revise their relationships in the a¢ nity network? Speci…cally, what are the incentives of agents to mend fences with enemies and transform hostile relationships into friendships? *Third*, how will any change in the a¢ nity network impact defensive collaborations in the alliance network? *Fourth*, and …nally, who are the agents that serve as "bridges" in the a¢ nity network to connect agents who would otherwise remain disconnected due to their mutual hostility?

We begin with a description of the architecture of the a¢ nity network H in the initial position. We assume that the distribution of positive and negative links is such that we can partition agents into non-empty *clusters* such that relationships within a cluster are friendly or neutral while relations across clusters are hostile or neutral. In the terminology of signed graphs following Davis (1967), a network with this particular distribution of positive and negative links is called *weakly balanced* (henceforth, simply *balanced*). In balanced a¢ nity networks, each cluster is composed of agents who are friends, friends of friends, friends of friends of friends... etcetera. Any links connecting agents across distinct clusters are always negative indicating that the agents are enemies. We assume that the partition of agents into clusters is a consequence of their disagreement over some norm. Agents within the same cluster subscribe to a common norm or core belief (for example, ideology, religion or politics) and thus any links that exist within the cluster are always friendly. Agents in distinct clusters di¤er in their perception of the norm and this dissonance implies that any links in H connecting an agent-pair from two separate clusters is always hostile. The norm or belief is captured by a scalar and thus permits classifying clusters as "close" or "distant" depending on the di¤erence between their adoptive norms or beliefs.

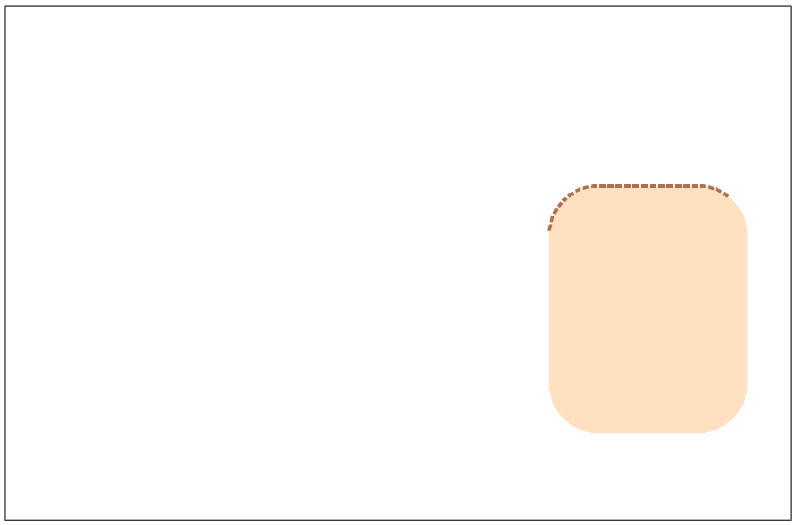

Figure 2: A Balanced A¢ nity Network

Figure 2 illustrates a balanced a¢ nity network with three clusters. Each agent $\boldsymbol{i}$ in H is indexed by a *friendship* measure, $_{\boldsymbol{i}}$, which is the number of $\boldsymbol{i}$'s friends minus the number of $\boldsymbol{i}$'s enemies. The *higher* the value of $_{\boldsymbol{i}}$, the more friends agent $\boldsymbol{i}$ has relative to enemies, and thus the *lower* is the level of hostility

the most hostility in H is the least connected in G with a neighborhood contained in the neighborhoods of all other agents.

Figure 4: Architecture of the Alliance Network

Once the alliance network is formed, we allow agents to revisit their relationships in the a¢ nity network. Therefore, we permit a two-way interaction between the a¢ nity and alliance networks. Once agents have formed su¢ cient alliances within their own cluster, then the ensuing gains from these links can provide an incentive for well-connected agents in two separate clusters to change an existing hostile relationship or a neutral one in H into a friendly one. This will be particularly true if the di¤erence between their perceived norms is su¢ ciently small. Of course, agents could also transform a neutral relationship into a hostile one but in our model there is no incentive to do so. Once these changes in the a¢ nity network are implemented, then this revised (and potentially unbalanced) a¢ nity network will spur a new round of alliances in the network G. Since new pathways of positive (friendly) links have been created across disparate pairs of clusters, two su¢ ciently well-connected agents from disparate clusters have an incentive to ally with each other. In particular, we show that *if two clusters are su¢ ciently close in their perceived norms, then su¢ ciently well-connected agents in the two clusters form a clique in the alliance network*. A *clique* in G is a set of agents such that every pair of agents in the sets are mutually linked. Thus, despite their dissonance over the norm, erstwhile hostile agents will have an incentive to ally if their disagreement over the norm is small.

Agents are assumed to belong to di¤erent clusters because of discord over what they believe should be the norm. The norm is captured by a scalar taking values over an interval $\_;\bar{}$, where $0 \quad \_ < \bar{} < 1$. Agents *within* a cluster $C\ (H_0)$, irrespective of whether they are neutral or friends in H_0, subscribe to a common norm $\ (H_0) \ 2\ \_;\bar{}$, and $\ (H_0) \ 6= \ _0(H_0)$ if $6= \ ^0$. This norm is assumed immutable and does not change. The greater the di¤erence, $j \ (H_0) \quad _0(H_0)j$, the more agents in clusters $C\ (H_0)$ and $C_0(H_0)$ di¤er in terms of core beliefs. We will de…ne for $\boldsymbol{i} \ 2\ C\ (H_0)$ and $\boldsymbol{j} \ 2\ C_0(H_0)$:

$$_{ij}(H_0) = \frac{1}{1 + j \ (H_0) \quad _0(H_0)j} \tag{1}$$

We will use $_{ij}(H_0)$ as a *measure of discord* between clusters and suppress reference to H_0 for brevity. If agents $\boldsymbol{i}$ and $\boldsymbol{j}$ belong to the same cluster, then $_{ij} = 1$ and there is no discord; if they belong to di¤erent clusters, then $_{ij} < 1$. Thus, the greater the dispersion in subscribed norms, the lower the value of $_{ij}$. Note that the measure of discord is a property of two clusters and not speci…cally of agents; in other words, for distinct agents $f\boldsymbol{i;j;k;l}g$ where $\boldsymbol{i;k} \ 2\ C\ (H_0)$ and $\boldsymbol{j;l} \ 2\ C_0(H_0)$, we have $_{ij} = \ _{kl}$. Also note that $_{ij} = \ _{ji}$. It is important to note once again that the discord between agents is …xed with respect to their position in H_0. Even if subsequently two agents $\boldsymbol{i}$ and $\boldsymbol{j}$ from di¤erent clusters establish a friendly relationship, their mutual discord $_{ij}$ is not equal to 1, i.e., they are still not in consonance with respect to their respective subscribed norms.

Agents will be permitted to make limited changes to the primitive H_0. A pair of agents $\boldsymbol{i}$ and $\boldsymbol{j}$ can change the relationship from neutral or enemy to friend, by each side incurring a cost that captures the e¤ort required to build the necessary trust. Thus, the formation of a friendly link requires *bilateral* consent of the pair of agents involved. The individual cost to agents $\boldsymbol{i}$ and $\boldsymbol{j}$ of converting $\boldsymbol{h_{ij}} \ 2\ f\ 1\boldsymbol{;}0g$ to $\boldsymbol{h_{ij}}$

and let $_{(0)}$ = 0 even if there are no isolated agents in G. The *degree partition* of G is denoted by D(G) = f$\boldsymbol{D}_0$(G)$;\boldsymbol{D}_1$(G)$;:::;\boldsymbol{D}_{\boldsymbol{m}}$(G)g, where all agents in the element $\boldsymbol{D}_{\boldsymbol{k}}$(G) of the partition have the same degree $_{(\boldsymbol{k})}$, $\boldsymbol{k}$ 2 f0$;$1$;:::;\boldsymbol{m}$g. The de…nition of path and connectedness are de…ned analogous to the case of signed networks. A maximally connected subnetwork G^0 in G is called a *component* of G. Given networks G and G^0, we will say that G is *denser* than G^0 if G^0 G. We will let G $\boldsymbol{ij}$ (respectively, G + $\boldsymbol{ij}$) denote the network obtained from G by deleting (respectively, adding) the link $\boldsymbol{ij}$.

An important network architecture that we will consider is a *nested split graph*. This network has the property that if $_{\boldsymbol{i}}$(G) $_{\boldsymbol{j}}$(G), then $N_{\boldsymbol{i}}$(G) $N_{\boldsymbol{j}}$(G) [f$\boldsymbol{j}$g. In other words, the neighborhood of a lower degree agent is contained within the neighborhood of a higher degree agent. Therefore, all allies of a less connected agent are also the allies of a more connected agent. Figure 3 illustrates this class of networks.

2.3 Gross Bene…ts from Alliances

Let Z denote the set of integers and consider the functions : Z_+ ! R_+ and $\boldsymbol{w}$: Z_+ ! R_+. The function $\boldsymbol{v}$ captures return from own degree while $\boldsymbol{w}$ captures the return from the partner's degree. Suppose agent $\boldsymbol{i}$ with degree $_{\boldsymbol{i}}$ forms a link with agent $\boldsymbol{j}$ with degree $_{\boldsymbol{j}}$. The incremental gross bene…t to agent $\boldsymbol{i}$ from this link with agent $\boldsymbol{j}$ depends on the degree of both agents involved and is assumed to be given by:

the region dramatically reducing operating and maintenance costs for its own fleet. Furthermore, there are indirect benefits from aligning with a higher degree node in G. The AUKUS military alliance came about as Australia was set to join a looser and more transactional military industrial agreement with France. Australia's post-cold war defense strategy concluded that "[Australia is] one of the most secure countries in the world... distant from the main centres of global military confrontation"[10]. Therefore a relatively inexpensive and limited agreement with France suited both. However, when Australia perceived China as a more present threat to Australia's homeland, the country reneged on the agreement with France and opted for AUKUS with the two most central nodes in NATO and the western alliance at approximately five times the monetary cost and incurring significant obligations on its military autonomy and sovereignty. The benefits of joining the tripartite AUKUS with stronger ties between and emanating from each node were, ceteris paribus, significantly greater than a bilateral agreement with France.[11]

Remark: (*Separable versus non-separable gross benefits*) We have postulated an additively separable in degrees specification for gross benefits in (2). Such a separable specification allows a transparent exposition of the main results. A non-separable formulation, in which gross benefit to agent i from a link with agent j is more generally specified as $_{i;\ j}$, would also yield the same set of results under appropriate conditions on . We demonstrate this in Section 6.3.

2.4 Cost of Hostility

Recall that agents can only form an alliance in G with those who are friends or distant friends in H and that an alliance requires mutual consent. By forming an alliance, an agent incurs a cost of linking which is a function of the hostility faced by the potential partner in H. Letting $c : Z \to R_+$ denote this linking cost function, we will impose the following assumptions on c.

Assumption (A.3): *For all* $\in Z$:
(a) $c(+ 1) < c( )$.
(b) $c( ) \quad c(+ 1) > c(\quad 1) \quad c( )$.

Therefore, the cost to an agent is lower when the potential partner faces less hostility. Further, the cost reduction realized with a higher friendship partner is greater than with a lower friendship partner. With a link, each agent assumes some of the risks posed by the hostile relationships of the potential partner. These risks are consequently lower if each partner has more friends and less enemies. This also explains

[10] Protection by Projection, *The Economist*, April 25, 2023.

[11] Another example substantiating assumption (A.2) is the Nordic fighter fleet agreements signed in spring 2023. The Nordic nations (Norway, Sweden, Denmark, and Finland), agreed to pool resources creating an integrated air defense. The Scandinavian peninsula is a great example of this effect. Because the shortest distance from Russian air bases to the allied coast is from the north, each nation has a relatively small geographic slice of detection zones, but their entire geography is collectively exposed around the clock. Therefore, the increased number of participants in resource-sharing greatly expands the benefit to each individual member through burden sharing in time, distance and capacity by eliminating duplications in detection, early warning, rotating alert and surge units, and command and control infrastructure. A similar dynamic is also observed in the NATO VJTF, a rotating multinational task force kept on ready alert to spearhead an immediate counter attack to an invasion. All NATO members rotate units through the VJTF to defend the collective's eastern perimeter.

why NATO does not permit admitting nations that are engaged in territorial disputes. A dispute indicates a high level of hostility towards that potential member and, therefore, a risk of con‡ict that commits the entire alliance. A more detailed look at this is NATO's admission of Finland, before the (presumptive) admission of Sweden. While both Finland and Sweden distrust Russia, only Sweden has some measure of hostile links with Turkey (also a NATO member). Finland, with its lower level of hostility, was therefore prioritized for admission to the military alliance (G) while the hostile link between Sweden and Turkey is being ‡ipped in H.

Each agent will also face some cost due to its own hostile relations. We will let $c_0 : Z \to R_+$ denote the cost to an agent from these hostile relations and assume that c_0 also satis…es conditions (a) and (b) of assumption A.3.

Remark: (*Costs that are functions of both hostility and degree*) The cost to agent i of linking with agent j is assumed to be $c(\ _j)$. It can be argued that this cost to agent i could be lower if j had greater degree, i.e., the cost function should be $c\ _j;\ _j$, with $c\ _j;\ _j + 1 < c\ _j;\ _j$. Our results would continue to obtain under this speci…cation as well as demonstrated in Section 6.3. We note, however, that this e¤ect is already captured in our basic model if we construe the net cost to agent $i \in C(H_0)$ from linking with agent j as:

$$e_i\ _j;\ _j = \begin{cases} c_0(\ _j) - v\ _j; & j = i \\ c(\ _j) - w\ _j; & j \in C(H_0) \\ c(\ _j) - {}_{ij}w\ _j - h^+_{ij}; & j \notin C(H_0) \end{cases} \tag{3}$$

Recalling assumption A.2(a), $e_i\ _j;\ _j$ is decreasing in $\ _j$.

Finally, we impose a joint restriction on the cost of forming an alliance in G and the cost of transforming a relationship in H. This assumption bounds the reduction in alliance costs that can be achieved with agents changing their a¢ nity relationship *within* a cluster from neutral to friendly. The logic is that agents are already friends or distant friends within a cluster. Thus, any reduction in alliance costs attained within a cluster by forming a more direct friendly relation is less than the cost of transforming an a¢ nity relationship. This shifts the impetus of agents to revise links *outside* rather than inside the cluster in an a¢ nity network.

Assumption (A.4): *For all* $\in Z$:

$$c_0(\ ) - c_0(\ + 1) <$$

Henceforth we will use the following notation to denote a unit increase in degree and friendship:

$$\begin{aligned} v(\ ) &\equiv v(\ + 1) - v(\ ) \\ c(\ ) &\equiv c(\ ) - c(\ + 1) \\ c_0(\ ) &\equiv c_0(\ ) - c_0(\ + 1) \end{aligned}$$

Note that we de…ne $c(\ )$ and $c_0(\ )$ such that they are positive due to A.3(a).

2.5 Payo¤s

In contrast to the contest function approach of the traditional literature, we adopt a reduced form additive speci…cation of payo¤s that re‡ect the tradeo¤s present in the model. There are essentially four factors at play: (i) the "economies of scale" from allying with those who have high degree in G; (ii) the cost of

cost of reaching out beyond their cluster. Speci…cally, if $\mathcal{G}$ (respectively, $\mathcal{G}$) is the stable alliance network under free riding (respectively, without free riding), then $\mathcal{G}$ $\mathcal{G}$.

3 Fixed A¢ nity Network

We begin our analysis with the case of a …xed a¢ nity network H_0. We then examine the implications of this …xed a¢ nity network on the topology of alliances in G. Therefore, we consider a one-way interaction between the a¢ nity and alliance networks. This section can be construed as a *short run* analysis when the horizon is su¢ ciently small for agents to e¤ect a change in relationships in the a¢ nity network. We are assuming here that relationships (whether friend or enemy) embodied in the network H_0 have taken time to coalesce. Within the time frame of the short run, new relations cannot be established in the a¢ nity network. The incremental utility to agent ***i*** from forging an alliance in G with a member ***j*** in its own cluster is given by:

$$_i(G + g_{ij}; H_0) \quad _i(G; H_0) = [v(_i(G) + 1) \quad v(_i(G))] + w \quad _j(G) + 1 \quad c(_j(H_0))$$

We will use a de…nition of stability inspired by Jackson and Wolinsky (1996).

De…nition (Pairwise-stability for monolayer networks): Given H_0, a network G is *pairwise-stable* if:

No agent $i \in N$ has an incentive to unilaterally delete an existing link with agent j in G , i.e., $_i(G ; H_0) \quad _i(G \quad g_{ij}; H_0) \quad 0$.

No pair of agents $i; j \in N$ who are unlinked in G g 0(()]TJ/F68 10156t25H

3.1 The Basic Link Formation Game

We will examine link formation in G through a dynamic game inspired by Aumann and Myerson (1988). The advantage of this approach is that it selects one among potentially multiple pairwise stable networks.[12]

pair of networks in the set and no improving path leading to a network outside the set. We will show below (Theorem 1) that a closed cycle is not possible in our link formation game. Thus, the only outcome is convergence to a limit network $G(H_0)$.

Theorem 1 *The basic link formation game converges to a limit network* $G(H_0)$ *which is a pairwise-stable network.*

Therefore, Theorem 1 also shows the *existence* of a pairwise-stable network. The proof is based on the fact that no agent has an incentive to delete a link that it formed along an improving path in G. With deletions of links ruled out, cycles cannot emerge along an improving path. Therefore, since the number of network architectures are …nite, the link formation game will converge to a pairwise-stable network.

Remark: (*Salient features of the basic game*) We note two facts about the dynamic game. *First*, we have the active agent deleting any unpro…table links and then proposing a new link to a passive agent. However, it is immaterial in our framework whether agents …rst delete links and then form a link, or …rst form a link and then delete links. This is because the incremental payo¤ from links that are formed will only increase by virtue of assumption A.2 as the degree of agents increase. Thus, as noted earlier, formed links are never subsequently deleted. *Second*, we allow the active agent to propose at most *one* link to a potential ally. We address in the next subsection the proposal of multiple links by an active agent.

3.2 The Pairwise-Stable Architecture of $G(H_0)$

Since friends and distant friends are contained within a cluster from assumption A.1, all alliances are between members of the same cluster. We will characterize the *intra*-cluster alliances formed by agents in G given H_0. Consider a given cluster $C(H_0)$, let $I = \{i_1; i_2; :::; i_n\}$ denote the set of agents arranged in increasing order of their index who belong to this cluster. Let $|C(H_0)|$ denote the size of this cluster. For ease of exposition, let us assume without loss of generality that:

$$i_1 \quad i_2 \quad i_3 \quad \quad i \tag{5}$$

with at l3336

The *friendship partition*, $\mathcal{C}(H_0) = \{\mathcal{C}_1(H_0); ...; \mathcal{C}_r(H_0); ...; \mathcal{C}_s(H_0)\}$, is the collection of friendship classes in $\mathcal{C}(H_0)$.

Friendship classes will play an important role in our characterization result. All agents within the same friendship class face the same level of hostility. Agents belonging to a lower-index friendship class face greater hostility than agents belonging to higher-index friendship class. Thus, for example, $i_i \in \mathcal{C}_1(H_0)$ and $i_n \in \mathcal{C}_s(H_0)$. Recalling the definition of a degree partition, let $\mathcal{D}(G) = \{D_0(G); D_1(G); ...; D_m(G)\}$ denote the *degree partition* of agents belonging to $\mathcal{C}(H_0)$ in the limit network $G \equiv G(H_0)$. We will now examine how agents are distributed across this degree partition as a function of their friendship measure, i.e., their hostility level. Specifically, we will connect $\mathcal{D}(G)$ to the friendship partition $\mathcal{C}(H_0)$.

We will begin by elaborating on how link formation proceeds according to our dynamic game. Recall that (active) agents proceed in increasing order of their index and can only propose alliances with those who belong to their cluster. Therefore, we can consider how links are formed within any given cluster, say $\mathcal{C}(H_0)$. The *first* active agent in $\mathcal{C}(H_0)$ to propose an alliance will be i_1. Note that at this stage, say $G_0(\ )$, no alliances have been formed and thus $\pi_i(G_0(\ )) = 0$ for all $i \in \mathcal{C}(H_0)$. Of course, if i_1 is agent 1 in cluster $\mathcal{C}_1(H_0)$ who initiates the game, then $G_0(\ ) = G^e$. The incremental payoff to agent i_1 from proposing an alliance with agent i_k is $\pi(0) + \omega(1) - c_{i_1} - c_{i_k}$. Since $c_{i_1} \leq c_{i_k}$ for all $i_k \in \mathcal{I} \setminus \{i_1; i_k\}$, it follows that the most profitable alliance is with agent i_n. However, if this alliance yields a negative payoff to at least one agent, then the link will not be formed. We can of course have isolated

paribus choose the one with lower enmity. Thus, at each stage of the link formation game, preferential attachment implies that an agent facing lower hostility will have at least as many alliances as an agent facing greater hostility.

Another way to visualize the intra-cluster NSG architecture is as a "core-periphery" subnetwork composed of a hierarchal order of agents according to their degree. The peripheral agents are those who are connected only to the core agents but not among themselves; the core agents are connected to all other core agents and di¤er only with respect to the peripheral agents they are connected to. Recall the degree partition $D(G) = \{\boldsymbol{D}_0(G); \boldsymbol{D}_1(G); ...; \boldsymbol{D}_{\boldsymbol{m}}(G)\}$ within the cluster $C(H_0)$ and let $\lfloor \boldsymbol{x} \rfloor$ denote the largest integer smaller than or equal to $\boldsymbol{x}$. The peripheral agents arranged in increasing number of alliances and their set of allies are as follows:

Table 1: Peripheral Agents in $C(H_0)$

Peripheral agents	Set of allies
$\boldsymbol{D}_1(G)$	$\boldsymbol{D}_{\boldsymbol{m}}(G)$
$\boldsymbol{D}_2(G)$	$\boldsymbol{D}_{\boldsymbol{m}}(G) \cup \boldsymbol{D}_{\boldsymbol{m}-1}(G)$
$\boldsymbol{D}_3(G)$	$\boldsymbol{D}_{\boldsymbol{m}}(G) \cup \boldsymbol{D}_{\boldsymbol{m}-1}(G) \cup \boldsymbol{D}_{\boldsymbol{m}-2}(G)$
$\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor}(G)$	$\boldsymbol{D}_{\boldsymbol{m}}(G) \cup \boldsymbol{D}_{\boldsymbol{m}-1}(G) \cup \boldsymbol{D}_{\boldsymbol{m}-2}(G) \cup \cdots \cup \boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor + 1}(G)$

The smallest set of peripheral agents are those in $\boldsymbol{D}_1(G)$ connected only to the core agents in $\boldsymbol{D}_{\boldsymbol{m}}(G)$ while the largest set of peripheral agents are those in $\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor}(G)$ who are connected to all core agents. The core agents arranged in decreasing number of alliances are as follows:

Table 2: Core Agents in $C(H_0)$

Core agents	Set of allies
$\boldsymbol{D}_{\boldsymbol{m}}(G)$	$\boldsymbol{D}_1(G) \cup \boldsymbol{D}_2(G) \cup \boldsymbol{D}_3(G) \cup \boldsymbol{D}_4(G) \cup \cdots \cup \boldsymbol{D}_{\boldsymbol{m}}(G)$
$\boldsymbol{D}_{\boldsymbol{m}-1}(G)$	$\boldsymbol{D}_2(G) \cup \boldsymbol{D}_3(G) \cup \boldsymbol{D}_4(G) \cup \cdots \cup \boldsymbol{D}_{\boldsymbol{m}}(G)$
$\boldsymbol{D}_{\boldsymbol{m}-2}(G)$	$\boldsymbol{D}_3(G) \cup \boldsymbol{D}_4(G) \cup \cdots \cup \boldsymbol{D}_{\boldsymbol{m}}(G)$
$\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor + 1}(G)$	$\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor}(G) \cup \boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor + 1}(G) \cup \cdots \cup \boldsymbol{D}_{\boldsymbol{m}}(G)$

The agents in $\boldsymbol{D}_{\boldsymbol{m}}(G)$ are core agents with the largest number of allies and they are connected to all agents – whether peripheral or core – in their cluster. Agents in $\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor + 1}(G)$ are core agents with the fewest number of allies and, while being connected to all core agents, are only allied with peripheral agents in the set $\boldsymbol{D}_{\lfloor \frac{m}{2} \rfloor}(G)$.

We now identify an interesting

4 Variable A¢ nity Network

So far we have kept the a¢ nity network as …xed and examined its in‡uence on the alliance network. However, it is possible that after harnessing su¢ cient economies of scale from their alliances, highly connected

4.1 The Augmented Link Formation Game

We now consider an augmented sequential link formation game that accommodates changes in both the a¢nity and alliance networks.

Given a non-empty primitive network H_0, link formation starts in the alliance network G starting from an empty network. The sequential process of link formation on this layer culminates in a limit network that we now denote as $G_1 \quad G(H_0)$.

The game now shifts to the network H. Players once again move sequentially in the order of their index starting from the state $H_0^{(0)} = H_0; G(H_0); 1$. The action set of the active agent in H is di¤erent from that in G. *First*, no links in H can be deleted. This is in accordance with our assumption that a¢nity relationships have matured bilaterally over a period of time and thus cannot be expunged unilaterally. *Second*, there is no incentive to convert a friend into an enemy because this makes an agent relatively unattractive as an ally to a potential partner. *Third*, by virtue of assumption (A.4), there is no incentive to revise a relationship within a cluster. An agent $\boldsymbol{i}$ can change an existing neutral relationship with agent $\boldsymbol{j}$ within the cluster to one of direct friends and the consequent increase in the friendship measure bestows a gain in own costs equal to $\boldsymbol{c}_0(\ _{\boldsymbol{i}}(H_0)) \quad \boldsymbol{c}_0(\ _{\boldsymbol{i}}(H_0)+1) < \ $. This is consonant with our formulation that any transformation of a¢nity links is a precursor to forging alliances in the alliance network, and two agents within the same cluster do not have to resort to this intermediate step in order to connect in G. Therefore, the only choice we allow an active agent is to commit resources to convert a hostile or neutral relation *outside* the cluster into a friendly one.

Suppose agent $\boldsymbol{i} \ 2 \ C\ (H_0)$ is the active agent. The active agent $\boldsymbol{i}$ can propose to an agent $\boldsymbol{j} \ 2 \ C^{\ 0}(H_0)$, $6= \ ^0$, with whom $\boldsymbol{h_{ij}} \ 2 \ f\ 1; 0g$ to change the relationship to a friend (i.e., to $\boldsymbol{h_{ij}} = +1$). Note that if another agent $\boldsymbol{k} \ 2 \ C\ (H_0)$ had prior to $\boldsymbol{i}$'s move already established a friendly relation with some agent, say $\boldsymbol{l}$, in $C^{\ 0}(H_0)$, then $\boldsymbol{i}$ has no incentive to make an overture to $\boldsymbol{j} \ 2 \ C^{\ 0}(H_0)$. This is because a friendly path between clusters $C\ (H_0)$ and $C^{\ 0}(H_0)$ has already been created in the a¢nity network through $\boldsymbol{h_{kl}} = +1$. Thus, agent $\boldsymbol{i}$ can free ride on this link to form alliances in G with members of $C^{\ 0}(H_0)$ without having to …rst transform an a¢nity link with $\boldsymbol{j} \ 2 \ C^{\ 0}(H_0)$.

Suppose, therefore, that when agent $\boldsymbol{i} \ 2 \ C\ (H_0)$ is the active agent and proposes to agent $\boldsymbol{j} \ 2 \ C^{\ 0}(H_0)$, then there is no friendly path connectiong clusters $C\ (H_0)$ and $C^{\ 0}(H_0)$. This new relationship in the a¢nity network imposes a cost of $\ > 0$ but increases $\boldsymbol{i}$'s friendship measure (lowers hostility level) to $\ _{\boldsymbol{i}}(H_0) + 1$. This increase in the friendship measure confers two bene…ts to agent $\boldsymbol{i}$. *First*, by virtue of assumption A.3(a), it decreases $\boldsymbol{i}$'s own costs:

$$\boldsymbol{c}_0(\ _{\boldsymbol{i}}(H_0)) = \boldsymbol{c}_0(\ _{\boldsymbol{i}}(H_0)) \quad \boldsymbol{c}_0(\ _{\boldsymbol{i}}(H_0)+1) > 0 \tag{9}$$

Second,

The active agent ***i*** chooses a potential partner ***j*** from another cluster with whom the sum of (9) and

Theorem 2 *The augmented link formation game converges to a limit* (G G(H); H H(G)) *which is pairwise-stable.*

4.2 The Pairwise-Stable multilayer Network

We now characterize the pairwise-stable multilayer network (G; H). Consider the augmented link transformation game when it moves from layer G_1 to layer H_1. Consider any two clusters $C(H_0)$ and $C'(H_0)$. Let $i \in C(H_0)$ and $j \in C'(H_0)$ be the most connected agents in their respective clusters (with the highest index agent chosen in case of a tie). Note from Proposition 2 that degree correlates positively with friendship, and thus these two agents are also the ones facing the lowest hostility in their respective clusters. Thus, as the following lemma indicates, these agents are the most likely candidates to transform their relationship to a friendly one since their realize the highest incremental utilities within their cluster from such a transformation in H and a subsequent alliance in G. Let $D(G_1) = \{D_0(G_1); D_1(G_1); ...; D_{m(\cdot)}(G_1)\}$ denote the degree partition of agents belonging to $C(H_0)$ in the network G_1 and define $D'(G_1)$ analogously. Also, let $s(\cdot)(H_0)$ (respectively, $s'(\cdot)(H_0)$) denote the highest friendship class in $C(H_0)$ (respectively, $C'(H_0)$). From Proposition 2 we know that $s(\cdot)(H_0) \in D_{m(\cdot)}(G_1)$ and $s'(\cdot)(H_0) \in D'_{m(\cdot)}(G_1)$.

Lemma 1 *Consider any two clusters* $C(H_0)$ *and* $C'(H_0)$ *and let* $i \in C(H_0) \setminus s(H_0)$ *and* $j \in C'(H_0) \setminus s'(H_0)$. *For any* $k \in C(H_0)$ *and* $l \in C'(H_0)$:

$$k(G_1 + g_{kl}; H_0 \quad h_{kl}) \quad k(G_1; H_0) \qquad i(G_1 + g_{ij}; H_0 \quad h_{ij}) \quad i(G_1; H_0)$$
$$l(G_1 + g_{kl}; H_0 \quad h_{kl}) \quad l(G_1; H_0) \qquad j(G_1 + g_{ij}; H_0 \quad h_{ij}) \quad j(G_1; H_0)$$

Let $\bar{\ }$ and $\bar{\ }'$ denote the respective degrees in G_1 of the maximally connected agents belonging to $C(H_0)$ and $C'(H_0)$, and $i(H_0)$ $\bar{\ }$ and $j(H_0)$ $\bar{\ }'$ denote their respective friendship levels. Further, let:

$$(G_1; H_0) \qquad \left[v(\bar{\ }) + c_0 \right]^{\bar{\ }} \tag{11}$$
$$'(G_1; H_0) \qquad \left[v(\bar{\ }') + c_0 \right]^{\bar{\ }'} \tag{12}$$

Then the maximally connected agent in $C(H_0)$ will propose to transform a neutral or hostile relationship with the maximally connected agent in $C'(H_0)$ if:

$$(G_1; H_0) + \quad _{ij} w \quad \bar{\ }$$

Now, recalling that $\omega(\ +1) > 0$ for all $\ \ 0$ by assumption A.2, let us de…ne:

$$_0(G_1; H_0) \quad \min \left\{ \frac{(G_1; H_0) + c^{\ -\ 0} + 1}{\omega^{\ -\ 0} + 1}; \frac{_0(G_1; H_0) + c^{\ -\ } + 1}{\omega(^{-}\ + 1)}; \right\} \tag{15}$$

$_0(G_1; H_0)$ is the threshold value of discord at which at least one agent, given their current degree in G_1 and friendship in H_0, is indi¤erent towards transforming a link in H_0. An examination of (15) shows that, ceteris paribus, two agents have an incentive to transform their relationship if its cost is low, the respective hostility levels they face is low (i..e, their -values are high), and link formation in G_1 has conferred a high enough degree on each to make it attractive to overcome any hurdle posed by their mutual discord. If $_{ij} = _{ji} < \ _0(G_1; H_0)$, then at least one agent will get a negative payo¤ from revising their a¢ nity relationship and will either not make such an overture (if it is the active agent) or will reject the overture (if it is the passive agent). From Lemma 1, this is also true for all pairs of agents drawn from the two clusters. Thus the existing a¢ nity relationships in H_0 between the two clusters will continue to remain hostile. Recalling (1), we have the following result:

Proposition 4 *Consider the a¢ nity network* H_0 *and suppose that for each pair of clusters* $C\ (H_0)$ *and* $C\ _0(H_0)$ *the divergence in their core norms satis…es*:

$$j\ \ (H_0) \quad _0(H_0)j > \frac{1}{_0(G_1; H_0)} \quad 1$$

Then $(G\ = G_1; H\ = H_0)$ *is the pairwise-stable multilayer network.*

If the dissonance in core beliefs is su¢ ciently large between each pair of clusters, then agents within a cluster have no incentive to change their cross-cluster a¢ nity relationships in H_0. Thus the architecture of H_0 remains unchanged. Consequently, the friendship levels of agents continue to remain the same as in H_0. When link formation returns to the alliance layer, then the strategic incentives to form alliances remain the same as when link formation …rst started in G. Since all pro…table opportunities to form alliances had already been exhausted in G_1, the architecture of the alliance network remains unchanged from G_1. Thus, all alliances that are forged continue to be within clusters and we do not observe any alliances spanning disparate clusters. Despite the high degrees of potential partners in G_1, and the accompanying economies of scale, all clusters continue to remain hostile and isolated in both the a¢ nity and alliance layers.

Once again consider $i\ 2\ C\ (H_0)$ and $j\ 2\ C\ _0(H_0)$ who are maximally connected in G_1 within their respective clusters. Now suppose $_{ij} = _{ji} > \ _0(G_1; H_0)$, i.e.,

$$j\ \ (H_0) \quad _0(H_0)j < \frac{1}{_0(G_1; H_0)} \quad 1$$

Then, because the di¤erence in their norms is relatively small, the two agents i and j have an incentive to transform their a¢ nity relationship. Therefore, there exists at least one agent pair in the two clusters who

will e¤ect a change in their a¢ nity relationship. Recall that *at most* one link between two clusters will be transformed into a positive one, since other agents in the two clusters can free ride on this "friendly" link to connect to others in the opposite cluster. Therefore, the pair transforming their link generate positive externalities for all other agents in the two clusters. Note that $H_1 \neq H_0$ because at least one neutral or hostile link in H_0 has been transformed to a friendly one. Since this transformation is predicated on the mutual pro…tability of an alliance in G, it follows that $G_2 \neq G_1$. Note an important di¤erence now from link formation in the very …rst iteration on G_0. In G_0, links could only be proposed to an agent *within* the cluster; in G_1, an active agent can now propose links to agent *outside* the cluster as well who are distant friends thus leading to cross-cluster alliances in G_2. Since degrees and friendships have now changed, accounting for these changes in the expressions for (11), (12) and (15) generates a new threshold value of

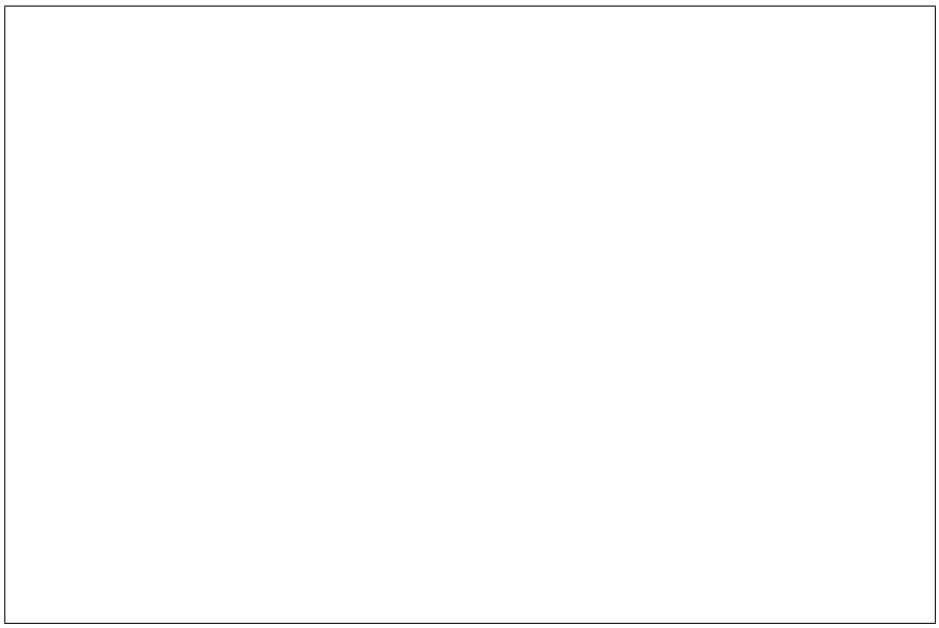

Figure 5: Inter-Cluster Clique in the Alliance Network

Part (b) of Proposition 5 states that if two agents from di¤erent clusters have formed an alliance, then all agents in these two clusters with greater degree and greater friendship measure will end up interconnecting with each other. The incenctives can best be explained with reference to Figure 5. Agent i_1 in cluster 1 and agent j_1 in cluster 2 have transformed their neutral relationship to friendly in the a¢ nity network which is indicated by the double line connecting the two agents. Note that $_{i_1} = _{j_1} = 1$ and $_{i_1} = _{j_1} = 1$ prior to transforming their relationship. Suppose i_1 was the active agent and j_1 was the passive agent when this relationship was transformed. Therefore, i_1's incremental payo¤ is:

$$[\quad v(1) + \quad c_0(1)] + [\quad _{12}w(2) \quad c(1)] \tag{16}$$

An identical expression holds for agent j_1. Now consider agent i_2 in cluster 1 who belongs to a higher friendship class than i_1 and also has greater degree. Then i_2 will also have a mutually pro…table link with j_1 in the alliance network. The incremental payo¤ to i_2 from forming an alliance with j_1 is:

$$v(3) + [\quad _{12}w(3) \quad c(2)] \tag{17}$$

Agent i_2 does not incur the cost since it can free ride on the friendship link between i_1 and j_1.[16] We can now compare term-wise the incremental payo¤s of i_1 and i_2. Note that $v(3) > v(1)$ from A.2(b), and $_{12}w(3) \quad c(2) > _{12}w(2) \quad c(1)$ by virtue of A.2(a) and A.3(a). Further, since $c_0(1) <$ from (A.4), it follows that (17) strictly exceeds (16). An identical argument establishes that j_1 will reciprocate the link with i_2, and that mutually pro…table links will also form betwTd[(Agen)28(t)]TJ/F63(et)28(w)28(Td[(Agen)28(t.01T

the transformed link $\boldsymbol{h}_{i_1 j_1} = +1$ in the a¢nity network spurs the creation of an inter-cluster clique in the alliance network composed of $\{\boldsymbol{i}_1; \boldsymbol{i}_2; \boldsymbol{j}_1; \boldsymbol{j}_2\}$. More generally, suppose $\_$ (respectively, $\_\,'$) be the agent with the lowest hostility level in $C\ (H_0)$ (respectively, $C\,'(H_0)$) who are willing to form an alliance with each other. Then all agents in $C\ (H_0)$ with -value exceeding $\_$, and all agents in $C\,'(H_0)$ with -value exceeding $\_\,'$, will also have a mutually pro…table alliance. Thus, we have an $(\ ;\ ^0)$-*clique* forming across two distinct clusters.

We now turn to the characterization of H . For each pair of clusters $C\ (H_0)$ and $C\,'(H_0)$, de…ne $'(G\ ;H\ )$ as in (15) but with respect to $(G\ ;H\ )$. We will let $H\ nH_0$ denote the new friendly relations that have been created and which did not exist in H_0.

Proposition 6 *Consider the limit network* H *in the pairwise-stable multilayer network* $(G\ ;H\ )$.
(a) *Two clusters* $C\ (H_0)$ *and* $C\,'(H_0)$ *are connected via a transformed friendly link in* $H\ nH_0$ *if*:

$$j\ \ (H_0)\ \ \ \ '(H_0)j < \frac{1}{'(G\ ;H\ )}\ \ 1 \tag{18}$$

(b) *Suppose two clusters* $C\ (H_0)$ *and* $C\,'(H_0)$ *have a mutually friendly link in* $H\ nH_0$ *and that* (*without loss of generality*) $\ (H_0) < \ '(H_0)$. *If there exists an "intermediate" cluster* $C\,''(H_0)$ *such that*:

$$(H_0) < \ \ ''(H_0) < \ \ '(H_0) \tag{19}$$

then both $C\ (H_0)$ *and* $C\,'(H_0)$ *have a friendly link with* $C\,''(H_0)$ *in* $H\ nH_0$.
(c) *Two clusters* $C\ (H_0)$ *and* $C\,'(H_0)$

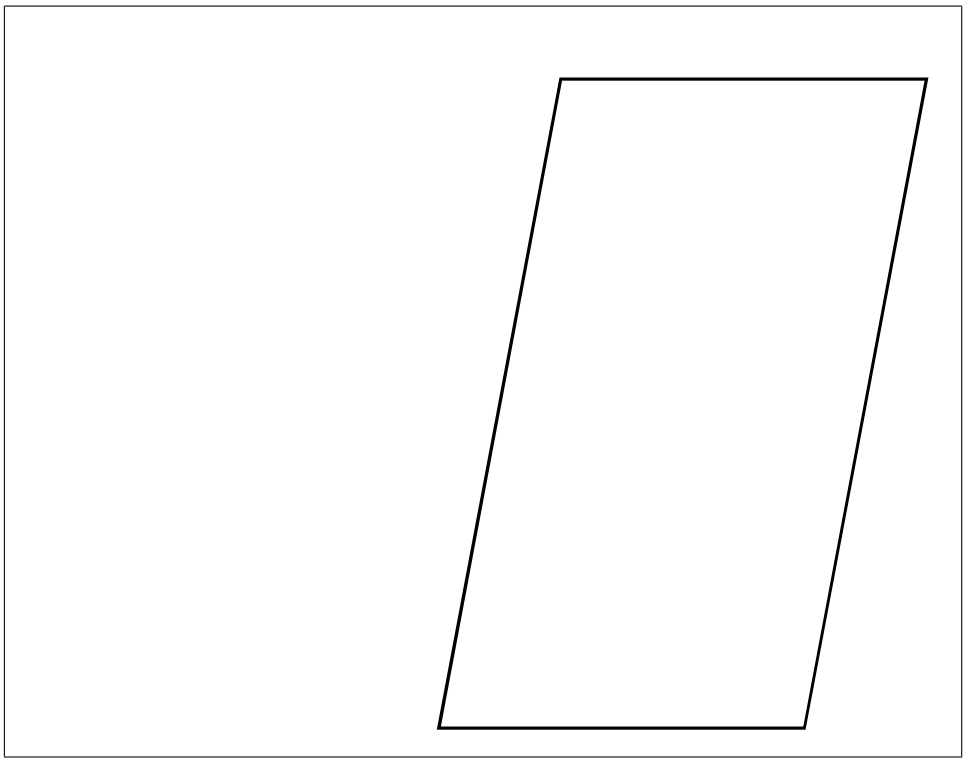

Figure 6: Overlapping Cliques in the Alliance Network

We now draw out the role of *bridge agents* who facilitate alliances across clusters. Consider Figure 6 which assumes that the relationship given by (18) holds between clusters 1 and 2, and between clusters 2 and 3. Further, cluster 2 is intermediate between the other two in the sense of (19). Finally, the relationship between clusters 1 and 3 is characterized by (20). The transformed relationship $\boldsymbol{h}_{\boldsymbol{i}_2\boldsymbol{j}_2} = +1$ (shown by the double line) in the a¢ nity network connects clusters 1 and 2 and prompts the creation of the inter-cluster clique f$\boldsymbol{i}_1; \boldsymbol{i}_2; \boldsymbol{j}_1; \boldsymbol{j}_2$g with agents $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ free riding on the friendly path created between the two clusters by $\boldsymbol{h}_{\boldsymbol{i}_2\boldsymbol{j}_2} = +1$. Likewise, the transformed relationship $\boldsymbol{h}_{\boldsymbol{j}_1\boldsymbol{k}} = +1$ precipitates the creation of the inter-cluster clique f$\boldsymbol{k}; \boldsymbol{j}_1$g; in this particular example, we are assuming that the degree and friendship measure of $\boldsymbol{j}_2$ is not su¢ cient for an alliance with $\boldsymbol{k}$ and the inclusion of $\boldsymbol{j}_2$ in the clique. Also, despite a friendly path now existing between clusters 1 and 3, the divergence in their core beliefs dissuades agents $\boldsymbol{k}$ and

are also in the intermediate cluster but are not members of the inter-cluster cliques due to high hostility in the a¢ nity network and low degree in the alliance network.

The more interesting case is when, despite the relationship between clusters 1 and 3 characterized by (20), the agents in these two clusters end up forming an alliance through the aegis of agents in cluster 2 who serve as bridge agents. Due to condition (20), agents such as $\boldsymbol{k}$ and $\boldsymbol{i}_1$ are su¢ ciently divergent in terms of their norms such that their incremental payo¤ does not cover the cost of transforming their relationship. However, with the friendly path that is now created through cluster 2, these agents can eschew the tranformation cost of and free ride to a mutually pro…table alliance in the a¢ nity network. Therefore, the links $\boldsymbol{h}_{\boldsymbol{i}_2\boldsymbol{j}_2} = +1$ and $\boldsymbol{h}_{\boldsymbol{j}_1\boldsymbol{k}} = +1$ confer positive externalities within clusters as well as across clusters permitting the formation of alliances between disparate agents who otherwise would not have an incentive to ally with each other. Therefore, through the bridge provided by agents $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$, we have an inter-cluster clique f$\boldsymbol{k}; \boldsymbol{i}_1; \boldsymbol{i}_2; \boldsymbol{j}_1; \boldsymbol{j}_2$g that spans three clusters.

5 Motivating Examples

We now provide a set of real world examples to substantiate our main results. To illustrate the *emergence of an NSG alliance structure among agents belonging to the same cluster in the a¢ nity network with high centrality* (*degree*) *corresponding to greater friendship measure*, we look to the Western Paci…c and its overlapping relationships in G in Figure 7. These relationships run largely through the United States in an NSG type con…guration. At the north end of the …gure, the NATO security alliance forms a connected alliance that runs dominantly through the United States to form paths to other states and alliances. Prior to 2023, the graph can be partitioned into a set of cliques (NATO, AUKUS, FIVE EYES) and an independent set (Japan, Republic of Korea, India, Taiwan, and Micronesia) forming a partial star architecture with the United States at its center. The United States' position as an economic partner, its cultural ties, and its role as a democratic security guarantor, imparts to it the highest aggregate ranking of friendship (lowest hostility). Thus, in accordance with our result, the United States has an exceptionally high centrality in the a¢ nity network and is by far the highest degree node in the alliance network.

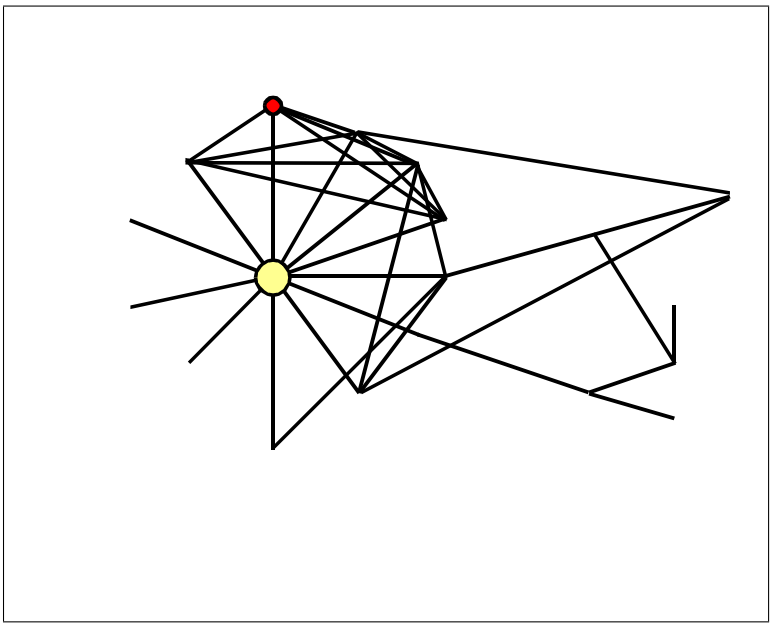

Figure 7: Alliance Network in the Paci…c

Our model indicates that *when two agents belong to separate clusters in the a¢ nity network, then as a precursor to forming a link in the alliance network this pair of agents have to establish a bridge in the a¢ nity network*. In the post-pandemic years (2022-23), Japan made moves in both the H and G layers. *First*, Japan and Australia signed a security cooperation agreement (a link in G) that built o¤ years of increasing economic ties (establishing a bridge in H). *Second*, Japan and South Korea are making signi…cant diplomatic and economic investments at the encouragement of the United States in ‡ipping their negative relationship into a positive one (establishing a bridge in H) as a precursor to security agreements (connecting in G). These moves are changing the existing star network in the Paci…c where the United States underwrote security for all states (the graph depicted in …gure 3(a)) into a more interconnected web of alliances (…gure 3(d)).

A real world application of our result that there is *alliance formation among enemies when their disagreement over a norm is relatively small* is provided by Balkan con‡ict from 1992-95 (Becker et al 2023). In this case, there were three (singleton) clusters, with each cluster corresponding to an ethnic group – Bosniak, Croat and Serb. Figure 5 depicts the multilayered links between the three ethnic groups indexed by layer. The period of the con‡ict (1992-95) can be divided into three periods, indexed by $\boldsymbol{t} \in \{1; 2; 3\}$, and the alliance structure that prevailed during these three periods is also indicated in …gure 5. At the multilayer network's most base *a¢ nity* layer, all relationships were negative since each group held deep and lasting animosities towards the others which predated the war by centuries. The other layers indicate

6.1 Con‡ict Models

We will show how the reduced form payo¤ function given by (4) can be deduced from an explicit con‡ict model. The con‡ict game is drawn from Baliga and Sjöstrom (2012). Consider agent ***i*** with neighborhood $N_i(G)$. Then agent ***i*** can …nd itself paired in pairwise con‡ict games –one with any of its own enemies, and one with any of the enemies of its allies. Therefore, agent ***i*** can be involved pairwise in $_i(G)$ con‡ict games. In each con‡ict game, the paired agents move simultaneously to choose either an aggressive action of *hawk* or a peaceful action of *dove*

Recalling (3), maximizing payo¤s in either the hawk-hawk or mixed strategy Nash equilibrium is equivalent to maximizing $_i$ (G; H_0) given by (4).[18] Therefore, our payo¤ speci…cation is grounded in a proper con‡ict game.

Remark: (*Other parametric restrictions*) Baliga and Sjöstrom (2012) classify agents as *opportunistic* if hawk (dove) is a best response to dove (hawk), and *hawk-dominant* (*dove-dominant*) if hawk (dove) is a dominant strategy. The dove-dominant case once again yields an empty a¢ nity network. The hawk-dominant case, and the pure strategy Nash equilibria of the opportunistic case yield total payo¤ given by (21). Thus, once again our reduced form payo¤ applies. The mixed strategy Nash equilibrium of the opportunistic case yields an anomalous result where agents would seek more hostilities and less allies.

6.2 Static Link Formation Game

We will explore a static alliance formation game adapted from Dutta et al. (1998) and show that the set of strongly stable equilibrium networks display the NSG architecture within each cluster. Thus, we provide an alternative approach to link formation that is distinct from the dynamic game as well as demonstrate the robustness of the NSG architecture. Each agent makes an announcement of intended alliances. An *announcement* by agent i is of the form $s_i = (a_{ij})_{j \in N \setminus i}$. The intended alliance $a_{ij} \in \{0, 1\}$, where $a_{ij} = 1$ means that i intends to form an alliance with j, while $a_{ij} = 0$ means that i intends no such alliance. Let S_i denote the set of announcements, or strategies, of agent i. An alliance between agents i and j is formed if and only if $a_{ij} = a_{ji} = 1$. We denote the formed link by $g_{ij} = 1$ and the absence of a link by $g_{ij} = 0$. A strategy pro…le $s = \{s_1, s_2, ..., s_n\}$, consisting of a strategy for each agent, induces a network $G(s)$. To simplify the notation we shall often omit the dependence of the network on the underlying strategy pro…le.

A strategy pro…le $s = \{s_1, s_2, ..., s_n\}$ is *Nash* if and only if $_i(G(s_i, s_{-i}), H_0) \geq _i(G(s'_i, s_{-i}), H_0)$, for all $s'_i \in S_i$ and for all $i \in N$, where s_{-i} is the strategy pro…le of all agents other than i

De…nition (Strong Stability): A network G is said to be *strongly stable* if for any coalition $\boldsymbol{S}$ and any G^0 that can be obtained from G through deviations by $\boldsymbol{S}$, $_i(G^0; H_0) > _i(G; H_0)$ for some $\boldsymbol{i} \in \boldsymbol{S}$ implies that $_j(G^0; H_0)$ $_j(G; H_0)$ for some $\boldsymbol{j} \in \boldsymbol{S}$.

The de…nition of strong stability that we employ is due to Dutta and Mutuswami (1997). According to their de…nition, if a network G is *not* strongly stable, then there exists a coalition $\boldsymbol{S}$ that can deviate to some network G^0 in which *all* members of $\boldsymbol{S}$ are strictly better o¤.

De…nition (Equilibrium Network): A network G is an *equilibrium network* if there is a Nash strategy pro…le supporting G, and the network G is strongly stable.

In our network setting, the only *unilateral* decision that an agent has is to sever alliances. The …rst property of an equilibrium network is, therefore, that no agent should have an incentive to delete any subset of its alliances. Note that forming an alliance is a *bilateral* decision requiring agreement by both agents. The second property of an equilibrium network states that, for any coalition, the member agents have no incentive to bilaterally form alliances that did not exist in the equilibrium network. The second property permits a re…nement of the set of Nash networks that satisfy the …rst property. The next result shows that all equilibrium networks display an intra-cluster NSG structure in which the neighborhood of an agent with a lower friendship measure is nested within the neighborhood of an agent with a higher friendship measure.

Proposition 7 *An equilibrium network exists. In an equilibrium network* G*, all agents belonging to the same cluster form an alliance with an NSG architecture such that if* $_j$ $_i$*, then* $N_i(G)$ $N_j(G) \cup \{j\}$.

6.3 Non-Separable Bene…ts and Costs

We had assumed additively separable bene…t and cost functions. This permitted us to avoid interaction between degrees of agents, or between degree and hostility. However, our results would continue to hold under a more general non-separable speci…cation with suitable restrictions on the interaction terms. We now spell out the precise set of restrictions that are needed. Suppose the gross bene…t to agent $\boldsymbol{i}$ from a link with agent $\boldsymbol{j}$ is more speci…ed as $_i; _j$. the function $: Z_+^2 \to R_+$ is assumed to satisfy the following conditions:

Assumption (A.2)*: *For all* $\boldsymbol{i}; \boldsymbol{j} \in N$:

(a) $_i + 1; _j$ $_i$

6.4 Endogenous A¢ nity Network and Norms

We have assumed that in the initial position the a¢ nity network is given. As a …rst step towards a microfounded a¢ nity network, we can assume that the a¢ nity network is initially empty, and draw upon the de…nitive analysis of Hiller (2017) to augment our link formation game with the prior formation of an a¢ nity network. Adapting Hiller, we can craft an *a¢ nity formation game* as follows. Let f $_1$( )$;$ $_2$( )$;:::;$ $_K$( )g denotes the distribution of norms that separates agents. The parameter captures the dimension on which the norm is based, i.e., 2 fculture, ideology, politics, securityg. Assume that n_k denotes the number of agents who subscribe to the norm $_k$( ) such that n_k 1, n_k 6= n_{k+1}, and $\sum_{k=1}^{K} n_k = N$. Each agent is endowed with a given intrinsic level of strength that is normalized to unity. An agent can augment this strength through positive connections in the a¢ nity network with agents *who share the same norm*. Formally, the strength gained by an agent i from establishing a positive connection with agent j in the a¢ nity network is equal to 1 if $_i$( ) = $_j$( ) and 0 otherwise.

Each agent simultaneously proposes positive (friendship) or negative (enemy) links to other agents in the

8 Appendix

Proof of Theorem 1: We will show that cycles cannot emerge along an improving path in G. Let $g_{ij} = 1$

while for agent j it follows from A.2(b), A.2(a) and A.3(a) that:

$$v_j(G^0 + g_{kj}) \quad v_j(G^0) \quad + \quad w_k(G^0 + g_{kj}) \quad c(k)$$
$$v_j(G + g_{ij}) \quad v_j(G) \quad + [w(i(G + g_{ij})) \quad c(i)] \quad 0$$

and at least one LHS is strictly positive. Therefore, agents k and j have a mutually pro…table link in G^0. ■

Proof of Proposition 2: We will …rst prove that $_1(H_0) \quad D_1(G_1)$. Suppose to the contrary that $i \; 2 \; _1(H_0) \setminus D_l(G_1)$ for $l \quad 2$. Thus, $_i(G_1) > \; _j(G_1)$ for $j \; 2 \; D_1(G_1)$. Let $k \; 2 \; N_i(G_1) \, n N_j(G_1)$ denote the agent with whom i formed a link when it had $_j(G_1)$ number of links, i.e, the same number of links as j. Let $G_0(\;)$ denote the stage along the improving path when this link was formed, and so $_j(G_1) = \; _i(G_0(\;))$. Therefore:

$$v(_i(G_0(\;))) + [w(_k(G_0(\;)) + 1) \quad c(k)] \qquad 0 \qquad (28)$$
$$v(_k(G_0(\;))) + [w(_i(G_0(\;)) + 1) \quad c(i)] \qquad 0 \qquad (29)$$

and at least one inequality is strict. Since $k \; 2 \; N_j(G_1)$ in the limit network G_1, it must be true that agents j and k do not have a mutually pro…table link in G_1:

$$\min \quad v(_k(G_1)) + \; w \quad _j(G_1) + 1 \quad c(j) \; ; \quad v \quad _j(G_1) \quad + [w(_k(G_1) + 1) \quad c(k)] < 0 \qquad (30)$$

However, since $_j \quad _i$ (given that $i \; 2 \quad _1$) and $G_0(\;) \quad G_1$, it follows from $_j(G_1) = \; _i(G_0(\;))$ and A.3(a) that:

$$v \quad _j(G_1) \quad + [w(_k(G_1) + 1) \quad c(k)] \qquad v(_i(G_0(\;))) + [w(_k(G_0(\;)) + 1) \quad c(k)] \quad 0$$
$$v(_k(G_1)) + \; w \quad _j(G_1) + 1 \quad c(j) \qquad v(_k(G_0(\;))) + [w(_i(G_0(\;)) + 1) \quad c(i)] \quad 0$$

which contradicts (30). Thus, $_i(G_1) \quad _j(G_1)$ for all $j \; 2 \; D_1(G_1)$ and $i \; 2 \quad _1(H_0)$, and hence $_1(H_0) \quad D_1(G_1)$. We now prove that $_s(H_0) \quad D_m(G_1)$. We have already shown in the main text that $i_n \; 2 \; _s(H_0)$ and $i_n \; 2 \; D_m(G_1)$. The same argument can be repeated for each member of

Thus $i^{(1)} \in N_i(G)$.

Now suppose this property is true for agents $i^{(1)}, i^{(2)}, \ldots, i^{(r)} \in C$, i.e., these agents are the first r partners of i_k and belong to $N_i(G) \setminus N_i(G)$. Consider the next partner $i^{(r+1)}$ of agent i_k and suppose this link was formed in stage $G_0^{(r)}$ along the improving path. Suppose i_k was the active player when this link was formed. Since $_i G_0^{(r)} = r_i G_0^{(r)}$, it follows from Proposition 1 that $i^{(r+1)}$ and i_l also have a mutually profitable link when i_l is the active agent. Now suppose $i^{(r+1)}$ was the active agent when the link with i_k was formed. Then, similar to the reasoning with $i^{(1)}$, agent $i^{(r+1)}$ would have first formed this link with i_l. Therefore, $i^{(r+1)} \in N_i(G)$. This completes the induction step and proves the nestedness property. ■

Proof of Theorem 2: In the augmented link formation game, as each iteration of link formation occurs in the alliance network G_r, $r \geq 1$, potentially new alliances are added but none of the existing links are deleted. Therefore, letting the vector $d(G) = \{d_1(G), d_2(G), \ldots, d_N(G)\}$ denote the *degree distribution* of [illegible] and $d(G_r)$

[illegible] $d(G$

The verification for agents j and l is identical. ■

Proof of Proposition 4: The proof is provided in the main text. ■

Proof of Proposition 5:

a. We have proved in Proposition 3 that G_1 has an NSG architecture in each cluster. Now suppose this is true for G_r, $r \geq 2$. We will prove it for G_{r+1} by contradiction. Suppose there exists a cluster $C(H_0)$ with agents i and j such that $\eta_i(G_{r+1}) \geq \eta_j(G_{r+1})$ but $N_i(G_{r+1}) \not\subset N_j(G_{r+1})$. In particular, there exists an agent $k \in C(H_0)$ such that $k \in N_i(G_{r+1}) \cap N_j(G_{r+1})$. Since $C(H_0)$ has an NSG structure in G_r, and $G_r \subset G_{r+1}$, the link $g_{ik} = 1$ must have been added when link formation was occurring in G_{r+1}. Thus, $\eta_i(G_r) < \eta_j(G_r)$. Recalling Proposition 3 which demonstrated that degree is positively correlated with friendship, it follows that $\eta_i \neq \eta_j$. Now suppose the network is $G_{r+1}(\ )$ when the link $g_{ik} = 1$ is formed in G_{r+1}. There are two possible cases.

Case I: Suppose i was the active agent and k acquiesced as the passive agent. Then, in some subsequent state $(G_{r+1}(\ )^0; H_r; k)$, i.e., when k is the active agent, then k will have a mutually profitable link with j.

$$v_k\left(G_{r+1}(\ )^0 + g_{kj}; H_r\right) - v_k\left(G_{r+1}(\ )^0; H_r\right) = v_k\left(G_{r+1}(\ )^0\right) + w_j\left(G_{r+1}(\ )^0 + 1\right) - c(\eta_j)$$

From A.2(b):

$$v_k\left(G_{r+1}(\ )^0\right) \leq v(\eta_k(G_{r+1}(\ )))$$

and, since $\eta_i(G_r) < \eta_j(G_{r+1}(\ )^0)$, from A.2(a) and A.3(a):

$$w_j\left(G_{r+1}(\ )^0 + 1\right) - c(\eta_j) > w(\eta_i(G_r) + 1) - c(\eta_i)$$

Therefore:

$$v_k\left(G_{r+1}(\ )^0 + g_{kj}; H_r\right) - v_k\left(G_{r+1}(\ )^0; H_r\right) > v(\eta_k(G_{r+1}(\ ))) + w(\eta_i(G_r) + 1) - c(\eta_i) \geq 0$$

where the second strict inequality follows from the fact that agent k had acquiesced to a link with i when the network was $G_{r+1}(\ )$. Agent j will reciprocate because:

$$\begin{aligned} v_j\left(G_{r+1}(\ )^0 + g_{kj}; H_r\right) - v_j\left(G_{r+1}(\ )^0; H_r\right) &= v_j\left(G_{r+1}(\ )^0\right) + w_k\left(G_{r+1}(\ )^0 + 1\right) - c(\eta_k) \\ &> v(\eta_i(G_r)) + [w(\eta_k(G_{r+1}(\ )) + 1) - c(\eta_k)] > 0 \end{aligned}$$

where the last strict inequality follows since i had proposed a link to k in $G_{r+1}(\ )$. Therefore, it cannot be the case that when all profitable opportunities have been exhausted in G_{r+1} then agents k and j will remain unlinked.

Case II: Suppose k was the active agent when the network was $G_{r+1}(\ )$. Then, according to the link

formation protocol, k would have proposed a link with agent j rather than i because:

$$\begin{aligned} {}_k(G_{r+1}(\ ) + g_{kj}; H_r) \quad {}_k(G_{r+1}(\ ); H_r) &= v({}_k(G_{r+1}(\ ))) + w \quad {}_j(G_{r+1}(\ )) + 1 \quad c({}_j) \\ &> v({}_k(G_{r+1}(\ ))) + [w({}_i(G_r) + 1) \quad c({}_i)] \end{aligned}$$

and once again agent j will accept the proposal. Therefore, once again we have a contradiction.

It follows that each intra-cluster architecture in G_{r+1} will have an NSG architecture. Since G is reached in a …nite number of steps, it follows that the result also holds for G .

b. Let $i \in C(H_0) \setminus {}_l(H)$ and $j \in C'(H_0) \setminus {}_{l'}(H)$. Let $H_{;0}$ denote the a¢ nity network H in which there is no friendly link between clusters $C(H_0)$ and $C'(H_0)$. There are two possible cases:

Case I: Suppose i and j incurred the cost of transforming their a¢ nity relationship allowing all other agents in the two clusters to free ride on the friendly path they have created. Following the same argument as Lemma 1, for any $k_1 \in C(H_0) \setminus \{i\}$ and $k_2 \in C'(H_0) \setminus \{j\}$:

$$\begin{aligned} {}_{k_1}G; H_{;0} \quad h_{k_1k_2} \quad {}_{k_1}G \quad g_{k_1k_2}; H_{;0} \quad {}_iG; H_{;0} \quad h_{ij} \quad {}_iG \quad g_{ij}; H_{;0} \quad 0 \\ {}_{k_2}G; H_{;0} \quad h_{k_1k_2} \quad {}_{k_2}G \quad g_{k_1k_2}; H_{;0} \quad {}_jG; H_{;0} \quad h_{ij} \quad {}_jG \quad g_{ij}; H_{;0} \quad 0 \end{aligned}$$

where at least one of the last inequality in each case is strictly positive. Since k_1 and k_2 free ride, it follows that:

$$\begin{aligned} {}_{k_1}(G; H) \quad {}_{k_1}(G \quad g_{k_1k_2}; H) > {}_{k_1}G; H_{;0} \quad h_{k_1k_2} \quad {}_{k_1}G \quad g_{k_1k_2}; H_{;0} > 0 \\ {}_{k_2}(G; H) \quad {}_{k_2}(G \quad g_{k_1k_2}; H) > {}_{k_2}G; H_{;0} \quad h_{k_1k_2} \quad {}_{k_2}G \quad g_{k_1k_2}; H_{;0} > 0 \end{aligned}$$

and the result follows.

Case II: Suppose a pair of agents, where at least one agent di¤ers from i or j, were the ones transforming their a¢ nity relationship. Call this pair of agents transforming their a¢ nity relationship as $f \in C(H_0) \setminus {}_s(H)$ and $g \in C'(H_0) \setminus {}_{s'}(H)$, where $s \neq l$ and $s' \neq l'$. Following the same argument as that in Case I, all agents with friendship measures greater than or equal to those of f and g will also have an incentive to form an alliance. Now consider agents i and j from the statement of the proposition. These two agents will free ride on the friendly link created by f and g and have a pro…table alliance by hypothesis. Thus, for any two agents k_1 and k_2 whose friendship measures are greater than or equal to those of i and j and who also free ride, we have:

$$\begin{aligned} {}_{k_1}(G; H) \quad {}_{k_1}(G \quad g_{k_1k_2}; H) \quad {}_i(G; H) \quad {}_i(G \quad g_{ij}; H) \quad 0 \\ {}_{k_2}(G; H) \quad {}_{k_2}(G \quad g_{k_1k_2}; H) \quad {}_j(G; H) \quad {}_j(G \quad g_{ij}; H) \quad 0 \end{aligned}$$

Proof of Proposition 6: The proof follows from the definition of the threshold value, $_0(G;H)$. ■

Proof of Proposition 7: To save space, we will suppress reference to H_0.
(*Existence*): We first establish existence. Recall that all alliances are formed within clusters. Consider the network in which each cluster C is complete, i.e., all agents in each cluster are mutually interconnected. Denote this network as G^c. If it is an equilibrium, then we are done. Otherwise, there exists a coalition S^0 and a network G^0 that can be obtained from G^c by S^0 such that $_i(G^0) > _i(G^c)$ for all $i \in S^0$. Since all alliances are intra-cluster, it implies that $S^0 \subseteq C$ for some cluster C. Specifically:

$$_i(G^0) = v\ _i\ G^0 \quad c_0(\ _i) + \sum_{j \in N(G^0)} w\ _j\ G^0 \quad c(\ _j) > \ _i(G^c); \quad i \in S^0$$

Since no new links could be added in G^c, the deviation must involve members in S^0 deleting their links. This implies in particular that in the cluster C:

$$v(|C| - 2) + [w(|C| - 1) - c(\ _j)] < 0; \quad i \in S^0; \quad j \in N_i(G^c) \cap N_i(G^0) \tag{31}$$

If G^0 is an equilibrium, then we are done. Otherwise, there exists a coalition S^{00} that can obtain a network G^{00} in which each member is strictly better off. We claim that this movement from G^0 to G^{00} can only involve a deletion of links. Suppose to the contrary that the movement from G^0 to G^{00} involves addition of links and let $S^0 \setminus S^{00}$ denote the non-empty subset of agents who are involved in forming alliances, either among themselves or with others in $S^{00} \cap S^0$ in the move from G^0 to G^{00}. Note that this intersection cannot be empty because firms in $C \cap S^0$ are completely connected among themselves; thus a member of S^0 has to be involved if new links are created starting from G^0. Consider any $i \in S^0 \setminus S^{00}$. Since i was completely connected in G^c, and deleted links in the move to G^0, any new alliance that it forms in the move to G^{00} must be with some agent $j \in N_i(G^c) \cap N_i(G^0)$ with whom it earlier dissolved an alliance. Since the deviation to G^{00} is strictly profitable:

$$v\ _i\ G^{00} \quad 1 + \ _j\ G^{00} \quad c(\ _j) > 0 \tag{32}$$

However, $_i(G^{00}) \quad 1 \quad |C| - 2$ and

in themove ton

and j respectively from forming a link in G^0 is equal to:

$$\sum_{l \in N_k(G^0)} \quad (35)$$

$$\sum_{l \in N_j(G^0)} \quad (36)$$

Note that $N_i(G) \quad N_k(G^0)$ and $N_j(G) \quad N_j(G^0)$. For all $l \in N_i(G)$, since $\quad _i$, it follows from respectively parts (b) and (d) of A.2 that $(\quad) \quad (\quad) \quad (\quad)$. Further, from A.2 (a), $\quad + 1 \quad + 1 \quad + 1 \quad + 1$. Further, since $\quad$, from A.3 (a), $c \quad + 1 \quad c \quad + 1$. Therefore, each term in (35) dominates the corresponding term in (33). Likewise, noting that $\quad _i$, each term in (36) dominates the corresponding term in (34). This proves the result. ■

Proof of Lemma 1 for the Non-separable Case: Dropping reference to G_1, we will let $\quad = \quad (G_1)$ and $\quad + 1 = \quad (G_1 + g_{ij})$. Then, $\quad (G_1 + g_{ij}; H_0 \quad h_{ij}) \quad \quad (G_1; H_0)$ is equal to:

$$\sum_{h \in N(G_1)} \quad (\quad) + [c(\quad) \quad c(\quad + 1; \quad + 1)] + \quad + 1 \quad + 1 \quad c \quad + 1$$

Similarly, $\quad (G_1 + g_{kl}; H_0 \quad h_{kl}) \quad \quad (G_1; H_0)$ is equal to:

$$\sum_{h \in N(G_1)} \quad (\quad) + [c(\quad) \quad c(\quad + 1; \quad + 1)] + [\quad (\quad + 1; \quad + 1) \quad c(\quad + 1)]$$

Since there is an NSG structure within each cluster with degree positively related to friendship, $N_k(G_1)$ $N_i(G_1)$ and thus $\quad _k$. It follows from parts (b) and (c) respectively of A.2 that:

$$\sum_{h \in N(G_1)} \quad (\quad) + [c \quad + 1)] + [c \quad + 1)]$$

and

terms:

$$c(\ _k;\ _i) \quad c(\ _k + 1;\ _i + 1) \quad c(\ _k;\ _k) \quad c(\ _k + 1;\ _k + 1) \tag{38}$$

From (37) and (38), it follows that:

$$c(\ _i;\ _i) \quad c(\ _i + 1;\ _i + 1) \quad c(\ _k;\ _k) \quad c(\ _k + 1;\ _k + 1)$$

Finally, note that $_{kl} = \ _{ij},\ _j \quad _l$ and $_j \quad _l$. Therefore, from A.2 (a):

$$_i + 1;\ _j + 1 \qquad _k + 1;\ _j + 1 \qquad (\ _k + 1;\ _l + 1)$$

and from A.3 (a):

$$c \quad _j;\ _j + 1 \quad c \quad _l;\ _j + 1 \quad c(\ _l;\ _l + 1)$$

It follows that:

$$_i(G_1 + g_{ij}; H_0 \quad h_{ij}) \quad _i(G_1; H_0) \quad _k(G_1 + g_{kl}; H_0 \quad h_{kl}) \quad _k(G_1; H_0)$$

The veri…cation for agents j and l is identical. ■

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