1 Introduction

Microeconomic theory has informed the design of many markets and other institutions. Many new mechanisms have been proposed to allocate resources in environments in which transfers are not used or are prohibited. These environments include the allocation and exchange of transplant organs, such as kidneys (Roth, Sönmez and Ünver, 2004); the allocation of school seats in Boston, New York City, Chicago, etc. (Abdulkadiroglu and Sönmez, 2003); and the allocation of dormitory rooms at US colleges (Abdulkadiroglu and Sönmez, 1999). The mechanisms used elicit ordinal preferences of participants.[1]

The central concerns in the development of allocation mechanisms are incentives and ef - ciency.[2] The literature focused on Pareto ef ciency: a social alternative is Pareto ef cient if there exists no other social alternative that makes everybody weakly better off and at least one individual better off.[3] Pareto ef ciency however is a weak ef ciency concept; while interpersonal utility comparisons are not needed for Pareto ef ciency, it only gives a lower bound for what can be achieved through desirable mechanisms. In consequence, welfare economics—starting with Bergson (1938), Samuelson (1947), and Arrow (1963)—have long looked at stronger ef ciency concepts requiring an ef cient outcome to be the maximum of a social ranking of outcomes; an idea later named as resoluteness.[4] For instance, Arrow (1963), pp. 36-37, discusses the partial ordering of outcomes given by Pareto dominance, and observes:

> But though the study of maximal alternatives is possibly a useful preliminary to the analysis of particular social welfare functions, it is hard to see how any policy recommendations can be based merely on a knowledge of maximal alternatives. There is no way of deciding which maximal alternative to decide on.

Our paper carries out the Bergson-Samuelson-Arrow's program of analyzing stronger welfare criteria to discrete mechanism design, in which continuous transfers are not allowed and there is a nite number of alternatives. We study a broad class of discrete environments, merely imposing a natural richness assumption on preference domains; richness is a substantially weakening of Arrovian universal domain assumption and it is satis ed in many practically and theoretically relevant economic domains such as voting for candidates or issues with universal strict prefer-

[1]In the context of deterministic mechanisms without transfers eliciting ordinal information is all we can do. In addition, eliciting ordinal preferences is considered simpler and more practical (see Bogomolnaia and Moulin, 2001).

[2]For instance, Bogolomania and Moulin (2004) write that "the central question of that literature is to characterize the set of ef cient and incentive compatible (strategy-proof) assignment mechanisms."

[3]Relatedly, constrained Pareto ef ciency is also studied, e.g., in the context of allocation of resources, stable (or fair) matchings that are not Pareto dominated by other stable (or fair) matchings.

[4]Resoluteness has been a standard property in social choice since its conception and its failure is at the core of the Condorcet paradox, see e.g. Black (1948) and Campbell and Kelly (2003

ences, matching, and allocation of discrete resources without compensating transfers; for earlier uses of the richness assumption we study see Pycia and Troyan (2019).

We analyze welfare criteria imposed on social choice functions and social welfare functions. For every profile of individual preference rankings, a social choice function (SCF) determines what unique alternative should be implemented, while social welfare function (SWF) determines a societal ranking of alternatives. Allowing for partial societal rankings, we can treat an SCF as an SWF in which the outcome of SCF is ranked above all other alternatives. [5] Following Arrow (1963), we say that an SWF is Arrovian if, and only if, it satisfies the standard resoluteness, (strong) Pareto, and independence-of-irrelevant-alternatives postulates. An SWF is resolute if it has a unique social maximum for every profile of preferences; in particular, every SCF is resolute. An SWF satisfies the (strong) Pareto postulate if two socially and Pareto-comparable matchings are ranked so that the Pareto-dominant matching is ranked above the Pareto-dominated one. An SWF satisfies the independence of irrelevant alternatives if, given any two profiles of preferences and any two alternatives that are socially comparable under both profiles, if all individuals rank the two alternatives in the same way under both profiles, then the social ranking of the two alternatives is the same under both profiles. When we want to highlight the positive rather than normative aspects of an SCF we refer to it as a mechanism; we allow here both Arrovian and not Arrovian SCFs. We call a mechanism efficient with respect to an SWF if, for every preference profiles, the resulting outcome is a maximum of the SWF.[6] We say that a mechanism is Arrovian efficient if it is efficient with respect to some Arrovian SWF. Finally, we say that a mechanism is strategy-proof if, for any reports by other individuals, reporting her true ranking leads to the mechanism outcome being weakly better for an individual than any other report.

We introduce a mild auditability requirement that says that, in order to falsify a proposed mechanism outcome, it is sufficient to verify pairwise comparison of individuals' preferences of the outcome with only one challenging alternative (the challenger). This auditability property is attractive as it allows to falsify the mechanism outcome with a limited amount of information and thus largely preserves the privacy of participants' private information. [7]

In Theorem 1, we show that Arrovian efficiency is equivalent to Pareto efficiency and auditability. In Theorem 2 we show that auditability implies non-bossiness of Satterthwaite and Sonnenschein (1981) and in general the reverse implication fails via an example. We prove that the conjunction of individual strategy-proofness and non-bossiness is equivalent to group strategy-proofness, which is in turn equivalent to monotonicity (Maskin, 1999) (Theorem 3). [8] We also

[5] For analysis of welfare with partial orderings, see e.g. see Sen (1970, 1999), Weymark (1984), and Curello and Sinander (2020).

[6] There is a rich social choice literature on the correspondence between choice and the maximum of the SWF ranking in the context of social choice (see below). This literature is interested in rationalizing social choice rather than the efficiency of mechanisms, and hence it talks about mechanisms “rationalized by an SWF” rather than “efficient with respect to an SWF.”

[7] For the literature on privacy in mechanism design see the recent survey Pai and Roth (2018).

[8] Analogous two equivalences were established earlier for object allocation, see Pápai (2000) and Takamiya (2001); our proof approach is different and simpler.

show that for Pareto ef cient mechanisms, either of these equivalent conditions implies Arrovian ef ciency.

We illustrate these results by applying them to characterizations in two canonical economic domains. In voting with the universal strict preference domain, our results immediately imply that Arrovian ef ciency and Pareto ef ciency are equivalent conditions for an individually strategy-proof mechanism as all mechanisms in the universal domain are non-bossy. In allocation of objects for individuals with unit demand who have strict preferences over the objects—often referred to as house allocation problems—our insights allow us to leverage the results of Pycia and Ünver (2017) to fully characterize the class of auditable and ef cient mechanisms as the class of *trading cycles mechanisms*. This characterization provides a no-transfer counterpart of Akbarpour and Li (2020) insight that classical auctions are the "credible" mechanisms in their sense.[9]

We further use this last characterization to show that *almost sequential dictatorships* a0 another alternative. In contrast, we rely on the more commonly used strong Pareto postulate in economics, in which an alternative is Pareto dominated as soon as all agents weakly prefer another alternative and at least one agent's preference ranking is strict.

Our paper also contributes to the literature on characterizations of dominant strategy mechanisms for house allocation. Ehlers (2002) characterizes group-strategy-proof and Pareto-ef cient mechanisms in a maximal domain of weak preferences for which such mechanisms exist and proves a general impossibility result for the domain of all weak preferences. [12] Note that our concept of partial social ranking is different from Ehlers' allowing only certain weak preferences over assigned houses; Ehlers' work is not concerned with social rankings of outcomes and we have equivalence classes for indifferences. Pycia and Ünver (2017) characterizes group-strategy-proof and Pareto-ef cient mechanisms in the standard domain of strict preferences and Root and Ahn (2020) characterize properties of these mechanisms allowing for constraints and providing a synthetic treatment of many social choice domains; see also Barberà (1983) and Pápai (2000) who laid the foundations for this line of research. Ma (1994) characterized the class of strategy-proof, individually rational, and Pareto-ef cient mechanisms, and his characterization has been extended by Pycia and Ünver (2017) and Tang and Zhang (2015) to richer single-unit demand, by Pápai (2007) to multi-unit demand models, and by Pycia (2016) to settings with network constraints.

Sequential dictatorships have not been studied extensively with unit demand for goods, although their special cases have been. In a *serial dictatorship* (also known as a *priority mechanism*), the same individual chooses next regardless of which house the current individual picks. Svensson (1994) formally introduced and studied serial dictatorships rst; Abdulkadiroglu and Sönmez (1998) studied a probabilistic version of them where the order of individuals is determined uniformly randomly; Svensson (1999) and Ergin (2000) characterized them using plausible axioms. Allowing for outside options, Pycia and Ünver (2007) characterized a subclass of sequential dictatorships different from serial dictatorships. With multiple-house demand under responsive preferences, Hat eld (2009) showed that sequential dictatorships are the only strategy-proof, non-bossy, and Pareto-ef cient mechanisms, and Pápai (2001) characterized the sequential dictatorships through the properties of strategy-proofness, non-bossiness, and citizen sovereignty (see also Klaus and Miyagawa, 2002). In a general model allowing both the cases with and without transfers, Pycia and Troyan (2019) showed that a broad class closely resembling sequential dictatorships are precisely the mechanisms that are strongly obviously strategy-proof in their sense; see also Li (2015) and Pycia (2019). For characterizations of random serial dictatorships in terms of incentives, ef ciency, and fairness see Liu and Pycia (2011) and Pycia and Troyan (2019). Root and Ahn (2020) characterize the constrained social choice domains in which generalized sequential dictatorships are the only group strategy-proof and Pareto-ef cient mechanisms. As an application of their general theorem, they characterize sequential dictatorships as the only mechanisms which are group strategy-proof and Pareto ef cient in the roommates problem.

[12] Most of the literature on house allocation—including our paper—is not affected by Ehlers' impossibility result because it analyzes environments in which individuals' preferences are strict.

2 Model

2.1 Environments

Let I be a set of *individuals* and A be a set of *social alternatives*. Each individual i has a preference relation over A (i.e., a complete, reflexive, and transitive binary relation) denoted by $\succsim_i$. We denote its strict (i.e., anti-symmetric) part by $\succ_i$ and indifference (i.e., symmetric) part by $\sim_i$. Let $\mathcal{P}_i$ be the domain of preference relations for individual i, and let $\mathcal{P}_J$ denote the Cartesian product $\prod_{i \in J} \mathcal{P}_i$ for any $J \subseteq I$. Any profile $\succsim = (\succsim_i)_{i \in I}$ from $\mathcal{P} = \mathcal{P}_I$ is called a *preference profile*. For every $\succsim \in \mathcal{P}$ and $J \subseteq I$, let $\succsim_J = (\succsim_i)_{i \in J} \in \mathcal{P}_J$ be the restriction of $\succsim$ to J. Suppose that for every individual there is an exogenous equivalence relation $\sim_i$ on alternative set A. We say that the domain $\mathcal{P}_i$ is *rich* if the following two conditions are satisfied:

1. If for any two alternatives a and b we have $a \sim_i b$, then for every $\succsim_i \in \mathcal{P}_i$ we have $a \sim_i b$.
2. If no alternatives in $A' \subseteq A$ are $\sim_i$-equivalent, then all strict preferences on A' belong to $\mathcal{P}_i$.

Thus, effectively, $\mathcal{P}_i$ is the universal strict preference domain respecting $\sim_i$-equivalence classes.[13] We say that the preference profile domain $\mathcal{P}$ is *rich* if $\mathcal{P}_i$ is a rich preference domain for every $i \in I$ and for any two alternatives a and b such that $a \sim_i b$ for every $i \in I$, $a = b$. The last condition eliminates redundancies in our description of the preferences over alternatives. For instance, in house allocation, each social alternative a is a matching between individuals and objects from some set and $a \sim_i b$ if, and only if, the object matched to i is the same under a and b. In the rest of the paper, we assume that $\mathcal{P}$ is a rich preference profile domain for a fixed equivalence relation profile $(\sim_i)_{i \in I}$.

Throughout the paper, we fix I and A, and thus, a problem is identified with its preference profile.

A *(direct) mechanism* or a *social choice function* (SCF) is a mapping $\varphi : \mathcal{P} \to A$ that assigns an alternative for every preference profile (or, equivalently, for every problem). We denote the outcome of mechanism φ for a preference profile $\succsim$ as $\varphi[\succsim]$.

We denote by P^S the set of strict partial orderings over alternatives, where a strict partial ordering is a binary relation that is anti-symmetric and transitive, but not necessarily complete. We refer to elements of P^S as *social rankings*. A *social welfare function* (SWF) $F : \mathcal{A} \to P^S$ maps individuals' preference profiles to social rankings. If an alternative a is ranked higher than some other alternative b under $F(\succsim)$, we denote this

a limited amount of information; one of the reasons this is an attractive feature of a mechanism is that it allows challenges that rely on relatively little information and largely preserve individuals' privacy.

A mechanism is individually strategy-proof if for every individual, she weakly prefers the

3 Equivalences

In this section, we study individually strategy-proof and Arrovian ef cient mechanisms and establish for them equivalence results involving Pareto ef ciency, auditability, group strategy-proofness and more technical properties of non-bossiness and monotonicity.

First, we characterize Arrovian ef ciency with the help of auditability. [15]

Theorem 1. *A mechanism is Arrovian ef cient if, and only if, it is Pareto ef cient and auditable.*

Second, auditability is a strictly stronger condition than non-bossiness, even for a Pareto ef - cient mechanism.

Theorem 2. *Any auditable mechanism is non-bossy. The converse does not hold – even for Pareto-ef cient mechanisms.*

Third, the conjunction of the two non-cooperative properties: individual strategy-proofness and non-bossiness is equivalent to either group strategy-proofness or monotonicity. [16]

Theorem 3. *The following three conditions are equivalent for a mechanism:*

1. *group strategy-proofness,*
2. *the conjunction of individually strategy-proofness and non-bossiness,*
3. *monotonicity.*

This result generalizes similar results due to Pápai (2000) and Takamiya (2001) for house allocation environments to our more general setting. Its proof is relegated to the appendix.

To illustrate the results and our concepts, let us look at the

and notice that

$$\begin{aligned} \psi[\succsim] &= \{(1,A),(2,B),(3,C)\}, \\ \psi[\succsim'] &= \{(1,A),(2,C),(3,B)\}. \end{aligned}$$

Mechanism ψ does not satisfy non-bossiness because from $\succsim$ to $\succsim'$ only 1's preference changes and her assignment does not change, and yet other individuals' assignments change (leading to different equivalence classes of alternatives for either individual 2 and 3).

Mechanism ψ does not satisfy Arrovian efficiency. Indeed, by way of contradiction assume that ψ is Arrovian efficient with respect to some Arrovian SWF Ψ. Then $\Psi(\succsim)$ ranks alternative $\psi[\succsim]$ above $\psi[\succsim']$, and $\Psi(\succsim')$ ranks $\psi[\succsim']$ above $\psi[\succsim]$. But, this violates IIA, a contradiction that shows that ψ is not Arrovian efficient.

Mechanism ψ does not satisfy auditability as we can contest the alternative $\psi[\succsim]$ with alternative $b = \psi[\succsim']$.

Mechanism ψ does not satisfy group strategy-proofness because the group $\{1,3\}$ can beneficially manipulate by reporting $\succsim'_{\{1,3\}}$ instead of $\succsim_{\{1,3\}}$ (noticing $\succsim$

rank a over a^{0}" throughout the proof). Note that Pareto ef ciency of j implies that conditions (i) and (ii) are consistent with each other, and hence, that the SWF F is well de ned.

By de nition, F satis es the Pareto postulate. Furthermore, F is transitive: if F (<) ranks a^1 above a^2 and it ranks a^2 above a^3, then it ranks a^1 above a^3. To see this: if one of these $a^{`}$ (for ` = 1, 2, 3) equals j [<], then it must be that a^1 = j [<], and the claim is proven. If none of the $a^{`}$ equals j [<], then individuals unanimously rank a^1 above a^2 and unanimously rank a^2 above a^3; we conclude that individuals unanimously rank a^1 above a^3, and thus, F (<) ranks a^1 above a^3 by construction.

It remains to check that F satis es IIA. Take two preference pro les $<^1$ and $<^2$ such that each individual ranks two alternatives, say a and a^{0}, in the same way under the two preference pro les. If the two alternatives are comparable under both F $<^1$ and above a

Corollary 3. In the universal strict preference domain, for an individually strategy-proof mechanism the following two conditions are equivalent:

- Pareto efficiency,
- Arrovian efficiency.

One direction of the corollary follows from Theorem 2 and then Theorem 1 because, in the universal strict preference domain, every mechanism is non-bossy, and the other direction was established in Theorem 1.

4.2 Incomplete and Complete SWFs in House Allocation

We now apply our results to house allocation problems. Formally, a *house allocation environment* consists of the set of individuals I and a set of houses H. A social alternative for this problem is a matching. To simplify the definition of a matching, we focus on environments in which $|H| \geq |I|$. To define a matching, let us start with a more general concept that we use frequently below. A submatching is an allocation of a subset of houses to a subset of individuals, such that no two different individuals get the same house. Formally, a *submatching* is a one-to-one function $s : J \to H$; where for $J \subseteq I$, using the standard function notation, we denote by $s(i)$ the assignment of individual $i \in J$ under s, and by $s^{-1}(H)$ the individual that got house $H \in s(J)$ under s. Let the remaining individuals in a round of the algorithm. We define a control-rights structure as a function of the submatching that is fixed: A *structure of control rights* is a collection of mappings

$$(k, b) = \left\{ (k_\sigma, b_\sigma) : \overline{H}_\sigma \to \overline{I}_\sigma \times \{\text{ownership, brokerage}\} \right\}_{\sigma \in \overline{A}} .$$

The functions k_σ of the control-rights structure tell us which unmatched individual controls any particular unmatched house at a submatching σ, where at σ is the terminology we use when some individuals and houses are already matched with respect to σ. Agent i *controls* house $H \in \overline{H}_\sigma$ at submatching σ when $k_\sigma(H) = i$. The type of control is determined by functions b_σ. We say that the individual $k_\sigma(H)$ owns H at σ if $b_\sigma(H)$ = ownership, and that the individual $k_\sigma(H)$ brokers H at σ if $b_\sigma(H)$ = brokerage. In the former case, we call the individual an *owner* and the controlled house an *owned house*. In the latter case, we use the terms *broker* and *brokered house*. Notice that each controlled (owned or brokered) house is unmatched at σ, and any unmatched house is controlled by some uniquely determined unmatched individual. We need to impose certain conditions on the control-rights structures to guarantee that the induced mechanisms are individually strategy-

The algorithm starts with empty submatching $s^0 = \emptyset$ and in each round $r = 1, 2, ...$ it matches some individuals with houses. By s^{r-1}, we denote the submatching of individuals matched before round r. If $s^{r-1} \in \overline{A}$, then the algorithm proceeds with the following three steps of round r:

Step 1 *Pointing.* Each house $H \in \overline{H_{s^{r-1}}}$ points to the individual who controls it at s^{r-1}. Each individual $i \in \overline{I_{s^{r-1}}}$ points to her most preferred outcome in $\overline{H_{s^{r-1}}}$.

Step 2(a) *Matching Simple Trading Cycles.* A cycle

$$H^1 \to i^1 \to H^2 \to ... H^n \to i^n \to H^1,$$

in which $n \in \{1, 2, ..\}$ and individuals $i^\ell \in \overline{I_{s^{r-1}}}$ point to houses $H^{\ell+1} \in \overline{H_{s^{r-1}}}$ and houses H^ℓ point to individuals i^ℓ (here $\ell = 1, ..., n$ and superscripts are added modulo n), is simple when at least one individual in the cycle is an owner. Each individual in each simple trading cycle is matched with the house she is pointing to.

Step 2(b) *Forcing Brokers to Downgrade Their Pointing.* If there are no simple trading cycles in the preceding Step 2(a), and only then we proceed as follows (otherwise we proceed to step 3).

- If there is a cycle in which a broker i points to a brokered house, and there is another broker or owner that points to this house, then we force broker i to point to her next choice and we return to Step 2(a).[20]
- Otherwise, we clear all trading cycles by matching each individual in each cycle with the house she is pointing to.

Step 3 Submatching s^r is defined as the union of s^{r-1} and the set of newly matched individual-house pairs. When all individuals or all houses are matched under s^r, then the algorithm terminates and gives matching s^r as its outcome.

One important feature of the TC mechanisms is that we can, without loss of generality, rule out the existence of brokers at some submatching s if there is a single owner at s. We formalize this property as a remark:

Remark 1. Pycia and Ünver (2017) *For every TC mechanism such that for some s there is only one owner and one broker, there is an equivalent TC mechanism such that at s there are no brokers and the same owner owns all houses.*

[20]Importantly, broker i is unique by R1.

Using Theorem 2 and Pycia and Ünver (2017

Denote

$$
\begin{aligned}
a^1 &= \varphi[\succsim^1] = \{(1,B),(2,C)\},\\
a^2 &= \varphi[\succsim^2] = \{(1,C),(2,B)\},\\
a^3 &= \varphi[\succsim^3] = \{(1,C),(2,A)\},\\
a^4 &= \varphi[\succsim^4] = \{(1,A),(2,C)\}.
\end{aligned}
$$

Now, if there is a complete SWF F such that φ is Arrovian efficient, then $F(\succsim^1)$ ranks a^1 above a^4, and by IIA, this implies that $F(\succsim)$ ranks a^1 above a^4. Similarly, $F(\succsim^2)$ ranks a^2 above a^1, and by IIA, this implies that $F(\succsim)$ ranks a^2 above a^1. Further, and again similarly, $F(\succsim^3)$ ranks a^3 above a^2, and by IIA, this implies that $F(\succsim)$ ranks a^3 above a^2. Finally, $F(\succsim^4)$ ranks a^4 above a^3, and by IIA, this implies that $F(\succsim)$ ranks a^4 above a^3. But then $F(\succsim)$ fails transitivity, showing that there does not exist a complete SWF with respect to which φ is efficient. QED

We will use this lemma to characterize individually strategy-proof and Arrovian efficient mechanisms for $|H| > |I|$; we will characterize this class of mechanisms for $|H| = |I|$ later. The resulting class consists of sequential dictatorships. Formally, a sequential dictatorship is a TTC mechanism ψ^k such that for every $s \in \overline{A}$ and $H, H' \in \overline{H_s}$, $k_H(s) = k_{H'}(s)$, i.e., an unmatched individual owns all unmatched houses at s. For notational convenience, we will represent each $k_H(\cdot)$ as $k(\cdot)$. Sequential dictatorships turn out to be the class of Arrovian-efficient and individually strategy-proof mechanisms for this case:

Theorem 4. *Suppose $|H| > |I|$. A mechanism is individually strategy-proof and Arrovian efficient with respect to a complete social welfare function if, and only if, it is a sequential dictatorship.*

Proof of Theorem 4. If $|I| = 1$, the theorem is trivially true. Suppose $|I| \geq 2$.

$(\Longrightarrow)$ Consider a mechanism φ that is individually strategy-proof and efficient with respect to a complete Arrovian welfare function. By Theorem 2 and Corollary 4, φ is a TC mechanism $\psi^{k,b}$.

Fix an arbitrary preference profile $\succsim \in P$. We claim that at any round r of the algorithm $\psi^{k,b}$, there is exactly one individual who controls all houses. We prove it in two steps. First, let us show that there cannot be two (or more) individuals who each own a house. By way of contradiction, suppose that some individual 1 controls house A and some other individual 2 controls house B in round r.

Suppose s is the submatching created by the TC algorithm for $\psi^{k,b}$ before round r at $\succsim$. Fix house $C \in \{A, B\}$ as an unmatched house at s. Consider four auxiliary preference profiles $\succsim^\ell$ that all share the following properties: (i) each individual matched under s ranks houses under $\succsim^\ell$, $\ell = 1, \ldots, 4$, in the same way they rank them under $\succsim$, (ii) each individual i unmatched at s and different from individuals 1 and 2 ranks a unique s-unmatched house $H_i \notin \{A, B, C\} \cup H_s$ as her rst choice (such a unique house exists as $jHj > jIj$), and (iii) individuals 1 and 2 each rank all houses other than A, B, C lower than A, B, C. In particular, the four pro les differ only in how individuals 1 and 2 rank houses A, B, C: the ranking of A, B, C is the same as in the four preference pro les from the proof of Lemma 1.Nothoice at(j)]TJ/F105 10.9091 Tf4 7675TJ -.72935 Td y(j)]TJ/F107 3.0091 Tf

Lemma 2. *Suppose that $|H| = |I| \geq 3$ and a TC mechanism is Arrovian efficient with respect to a complete SWF. Then in this mechanism one individual cannot control a house while two others each own a house.*

Proof. Consider a TC mechanism ψ in which individual 1 owns house A, individual 2 owns house B, and individual 3 controls house C. We will show that there is no complete SWF such that ψ is Arrovian efficient. Consider the preference profile

$\succsim =$

1	2	3
B	C	A
C	A	B
A	B	C
⋮	⋮	⋮

and the following three additional preference profiles

$\succsim^1 =$

1	2	3
B	C	B
C	⋮	⋮
A		
⋮		

$\succsim^2 =$

1	2	3
C	C	A
⋮	A	⋮
	B	
	⋮	

$\succsim^3 =$

1	2	3
B	A	A
⋮	⋮	B
		C
		⋮

Regardless of whether individual 3 owns or brokers house C, we have

$$a^1 = \psi[\succsim^1] = \{(1,A),(2,C),(3,B)\};$$
$$a^2 = \psi[\succsim^2] = \{(1,C),(2,B),(3,A)\};$$
$$a^3 = \psi[\succsim^3] = \{(1,B),(2,A),(3,C)\}.$$

If there is a complete SWF F such that ψ is Arrovian efficient, then $F(\succsim^1)$ ranks a^1 above a^3, and by IIA, this implies that $F(\succsim)$ ranks a^1 above a^3. Similarly, $F(\succsim^2)$ ranks a^2 above a^1, and by IIA, this implies that $F(\succsim)$ ranks a^2 above a^1. Further, and again similarly, $F(\succsim^3)$ ranks a^3 above a^2, and by IIA, this implies that $F(\succsim)$ ranks a^3 above a^2. Then $F(\succsim)$ fails transitivity, showing that there does not exist a complete SWF with respect to which ψ is efficient. QED

Lemma 3. *Suppose that $|H| = |I| \geq 3$ and a TC mechanism is Arrovian efficient with respect to a complete SWF. Then, in any round of the TC algorithm, there is at most one broker.*

Proof. By way of contradiction, suppose that in some round of the TC mechanism there are more than one broker and let ψ be the continuation TC mechanism from this round onwards. Without loss of generality, in ψ individual 1 brokers house A, individual 2 brokers house B, and individual

3 brokers house C. We will show that there is no complete SWF such that j is Arrovian ef cient. Consider the following preference pro les

$$< = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & B & C \\ A & A & B \\ C & C & A \\ \vdots & \vdots & \vdots \end{array}.$$

and

$$<^1 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline A & B & C \\ C & A & B \\ \vdots & \vdots & \vdots \end{array}, \quad <^2 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & B & C \\ A & C & A \\ \vdots & \vdots & \vdots \end{array}, \quad <^3 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & A & B \\ C & C & A \\ \vdots & \vdots & \vdots \end{array}.$$

Denote

$$a^1 = j\,[<^1] = f\,(1,A)\,,(2,B)\,,(3,C)\,g\,;$$
$$a^2 = j\,[<^2] = f\,(1,B)\,,(2,C)\,,(3,A)\,g\,;$$
$$a^3 = j\,[<^3] = f\,(1,C)\,,(2,A)\,,(3,B)\,g\,.$$

If there is a complete SWF F such that j is Arrovian ef cient, then F $<^1$ ranks a^1 above a^3, and by IIA, this implies that F $(<)$ ranks a^1 above a^3. Similarly, F $<^2$ ranks a^2 above a^1, and by IIA, this implies that F $(<)$ ranks a^2 above a^1. Further, again similarly, F $<^3$ ranks a^3 above a^2, and by IIA, this implies that F $(<)$ ranks a^3 above a^2. Then F $(<)$ fails transitivity, showing that there does not exist a complete SWF with respect to which j is ef cient. QED

Proof of Theorem 5. If $jHj > jIj$, it follows from Theorem 4 and if $jHj = jIj = 1$, the theorem is trivially true. Hence, suppose $jHj = jIj > 1$.

(=)) Consider a mechanism j that is individually strategy-proof and ef cient with respect to a complete Arrovian welfare function. By Theorem 2 and Corollary 4, j is a TC mechanism $y^{k,b}$.

Fix < 2 P. We claim that at any round r of the algorithm for $y^{k,b}$, there is exactly one individual who controls all houses whenever $j\overline{I}_s j > 2$. We prove it in three steps (in accordance with Lemmas 1-3). Let s be the submatching created by the algorithm $y^{k,b}$ before round r for $<$.

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the proof of Lemma 3 above. Notice that

$$\psi^{k,b}[\succsim^\ell] = s \cup a^\ell,$$

where a^ℓs are defined as in the proof of Lemma 3 above. Furthermore, the same argument we used in the proof of Lemma 3 shows that there can be no SWF that ranks all three a^ℓs, is transitive, and satisfies IIA. Hence, there is no complete SWF that makes $\psi^{k,b}$ efficient, a contradiction.

Thus, a single individual owns all houses at round r when s is fixed for $|\bar{I}_s| > 2$ (by Corollary 4 and Remark 1).

This means that $\psi^{k,b}$ is an almost sequential dictatorship, as all TC mechanisms restricted to only two individuals are almost sequential dictatorships.

($\Leftarrow$) Consider an almost sequential dictatorship ψ^k. If ψ^k is a sequential dictatorship, then the proof of Theorem 4 works. So suppose it is not a sequential dictatorship. Hence, $|H| = |I|$. We construct a complete SWF $\mathcal{F}$ such that ψ^k is efficient with respect to $\mathcal{F}$. Under $\mathcal{F}$ any two matchings are ranked according to the preference relation of the first-round dictator; if she is indifferent, then the matchings are ranked according to the preference relation of the second-round dictator, etc., until only two individuals remain unmatched. At this round let 1 and 2 be the two individuals and A and B be the two houses remaining unmatched. Observe that there are only two matchings, a and b, in which all individuals' assignments are the same but the last two: in one 1 gets A and 2 gets B, and in the other vice versa. Then one of these two matchings is equal to $\psi^k[\succsim^0]$, where $\succsim^0$ ranks the assignment of any individual other than 1 and 2 in a (or equivalently b) as her first choice, and for 1 and 2, the new preferences are the same as the original ones under $\succsim$. We rank $\psi^k[\succsim^0] \in \{a, b\}$ before the other one under $\mathcal{F}(\succsim)$.

Formally, for every $a \in \mathcal{A}$, let sequential dictators $i_1, \ldots, i_{|I|-2}$ be defined as $i_1 = \kappa_H(\emptyset)$ for every $H \in \mathcal{H}$, and in general, $i_\ell = \kappa_H(\{(i_1, a(i_1)), \ldots, (i_{\ell-1}, a(i_{\ell-1}))\})$ for every $H \in \mathcal{H} \setminus \{a(i_1), \ldots a(i_{\ell-1})\}$ and $\ell = 1, \ldots, k$; then for every $b \in \mathcal{A} \setminus \{a\}$, we say $a\, \mathcal{F}(\succsim)\, b$ if one of the following two conditions holds:

1. there exists $k \in \{1, \ldots, |I| - 2\}$ such that $a(i_1) = b(i_1)$, ..., $a(i_{k-1}) = b(i_{k-1})$, and $a(i_k) \succ_{i_k} b(i_k)$;

   or

2. for every $\ell \in \{1, \ldots, |I| - 2\}$, $a(i_\ell) = b(i_\ell)$, and for $\succsim^0 \in \mathcal{P}$ where each i_ℓ ranks $a(i_\ell)$ first while the remaining two individuals have the same preferences as in $\succsim$, we have $\psi^k[\succsim^0] = a$.

By construction, $\mathcal{F}$ is complete, antisymmetric, and transitive. Moreover, it satisfies the Pareto postulate. To see that it also satisfies IIA, consider two distinct matchings, $a, b \in \mathcal{A}$, and $\succsim \in \mathcal{P}$ such that $a\, \mathcal{F}(\succsim)\, b$. Also consider another profile $\hat{\succsim} \in \mathcal{P}$ such that each individual i's preference over the two matching assignments is the same in $\hat{<}_i$ as in $<_i$. If a F $(<)$ b because of condition 1 above, then condition 1 continues to hold for $\hat{<}$ and thus a F $(\hat{<})$ b. On the other hand, if a F $(<)$ b because of condition 2 above, then a and b only differ in how the last two individuals are assigned the remaining two houses. Hence, the pro le constructed to check condition 2 for a F $(\hat{<})$ b, which we refer to as $\hat{<}^{0}$, would lead to $y^k[\hat{<}^{0}] = a$ because:

1. the rst jIj 2 dictators would still get their a assignments in the rst jI ` 2 rounds of the TC algorithm for $y^k[\hat{<}^{0}]$, and
2. thus, the assignment of remaining two individuals under $y^k[\hat{<}^{0}]$ would be identical with that under a as the relative ranking of their assignments under a and b are identical both in $<$ and $\hat{<}$, and the ranking of the other houses do not matter for nding the outcome of the almost serial dictatorship.

Thus, a F $(\hat{<})$ b, showing F satis es IIA. QED

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A Omitted Proof

Proof of Theorem 3. (Group strategy-proofness $\Rightarrow$ individual strategy-proofness and non-bossiness) By de nition, any group strategy-proof mechanism is immune to all single-person group deviations, and hence, it is also individually strategy-proof. To the contrary to the claim, suppose a group strategy-proof mechanism φ is not non-bossy. Then there exists some individual i, preference pro le $\succsim$, and i's preference relation $\succsim'_i$ such that $a = \varphi[\succsim]_i$ $\varphi[\succsim'_i, \succsim_{-i}] = a'$ and yet there exists some individual $j \neq i$ such that $a \neq$ k50(to)-2570i, 1 jjTd [3.006 1.918 Td [(a)]TJ/F42 8.3049 Tf 5.08 3.96 Td [(0)]TJ/F70 10.907

(Monotonicity $\Rightarrow$ group-strategy-proofness). Let φ be a monotonic mechanism. Consider a preference profile $\succsim$, a group $J \subseteq I$, and a possible deviation $\succsim'_J$. Suppose $a' = \varphi[\succsim'_J, \succsim_{-J}] \succsim_j \varphi[\succsim] = a$ for every $j \in J$ and for some individual $i \in J$ the preference relation is strict. Consider the preference profile of J, $\succsim_J$ such that a' is ranked higher than a and every other equivalence class of alternatives are ranked below these two alternatives' equivalence classes. $(\succsim_J, \succsim_{-J})$ is a φ-monotonic transformation of $\succsim$, and hence, $\varphi[\succsim_J, \succsim_{-J}] \sim_j a$ for all $j \in I$ by monotonicity of φ. Since a' is the top alternative in $\succsim_j$ for every $j \in J$ and $\varphi[\succsim'_J, \succsim_{-J}] = a'$, $(\succsim_J, \succsim_{-J})$ is also φ-monotonic transformation of $(\succsim'_J, \succsim_{-J})$, and hence, $\varphi[\succsim_J, \succsim_{-J}] \sim_j a'$ for every $j \in I$ by monotonicity of φ. Since $a \not\sim_i a'$, we obtain a contradiction. Thus, φ is group strategy-proof. QED

B An Incomplete Arrovian Social Welfare Function

The following example illustrates an incomplete Arrovian SWF.

Example 3: Consider a society (or an employer) assigning one task to each of three employees. All the tasks need to be completed, and the society would like to respect the preferences of the employees in assigning the tasks as much as possible. As a second order concern, the society would like to avoid assigning Task A to employee 1 (e.g. because of a belief that employee 1 is not very good in doing this job). The society thus has an SWF that has the maximum at a Pareto-efficient matching that does not assign Task A to employee 1 if there exists at least one Pareto-efficient matching that does not assign Task A to employee 1.

The society's SWF can be equivalently described in terms of a Trading Cycles mechanism ψ in which employee 1 brokers A, employee 2 has ownership of B and employee 3 has ownership of C: for any preference profile $\succsim_{\{1,2,3\}}$, the SWF $\Psi(\succsim)$ ranks any two distinct matchings a awenivale50 if Td ose

$\{(1,B),(2,A),(3,C)\}$

$\{(1,A),(2,C),(3,B)\}$ $\{(1,A),(2,B),(3,C)\}$ $\{(1,C),(2,A),(3,B)\}$

$\{(1,B),(2,C),(3,$

Figure 1: $Y(<)$ *in Example 3. For matching* a, b*, we have* $a\ Y(<)\ b$ *if and only if there is a directed path from* a *to* b *in this graph.*

■